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Fixed point and ordered metric spaces a review

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Abstract

we are going o discuss the development in fixed point theory in the field of ordered metric space *in this review paper*

Keywords. Fixed point , ordered metric spaces, partial ordered set, weakly increasing mapping.

Introduction

Upto 1890 it was assumed that the evolution of fixed point theory is in the method of successive approximations which was helpful in finding the solutions of differential equations by Charles Emile Picard. But by the beginning of twentieth century it was considered as an important part of analysis. However, from historical point of view, it is assumed that the major classical result in fixed point theory was given by d to L. E. J. Brouwer in 1912.However the publication of Banach Principle by great Polish Mathematician Stefan Banach gives an effective way to find the fixed point of a map.It gives realities and uniqueness for a self mapping in a metric space in which every cauchy sequence is converges to some point in the given matrix. This all has been comprehensively contemplated and summed up in numerous ways from [1 - 4].

It was Caristi who in 1976 introduced the concept of ordered metric space. He in his paper defined an ordered relation and proved a theorem by using a functional satisfying some specific conditions in a metric space.

Presence of fixed points has been considered as of late in [5], and a few speculations of the aftereffect of [5] are given in [6– 10]. Likewise, in [5] a few applications to framework conditions are introduced in [7, 8].A few applications to intermittent limit esteem issue and to some specific issues areseparately given. In 2008, O'Regan and Petrusel [10] gave some existence results for solving Fredholm and Volterraintegral equations. In a portion of the above works, the fixed point results are given for non decreasing functions. In 2010, I Altun and H. Simsek [11] presented some fixed point results in ordered metric spaces for non decreasing and weakly increasing functions.They gave an existence theorem for solution of two integral equations.In 2013, Poonam Kumam et al [12] introduced some fixed point theorems using non linear contraction conditions which are helpful in solving an integral equation.

Recently A R. Butt et al [13],have given some fixed point results which have been observed to be applicable in homotopy.

Definition and Preliminaries.

We begin this section by some notations, definitions and theorems required in our subsequent discussions.

Definition ([14]) : Let A be a non-empty set. Then $a \in A$ is called a fixed point of a mapping $g: A \to A$ if $g(a) = a$

Definition ([13]) : Consider a set B such that $B \neq \phi$. Then $(B, d, \preceq)$ is called an ordered (partial) metric space if

(i)$(B, \preceq)$ is a partially ordered set and (ii) (B, d) is a metric space.

Definition:A partially ordered set or poset is a set P and a binaryrelation $\leq$, such that for all $a, b, c \in P$

1 a ≤ a (reflexivity).

2 a ≤ b and b ≤ c implies a ≤ c (transitivity).

3 a ≤ b and b ≤ a implies a = b. (anti-symmetry).

Definitionsee [15,16]: Let (Z, ⪯) be a partially ordered set. Two functionsP,Q:Z→Z are said to be weakly increasing if Pz⪯ QPz and Qz⪯ PQz for all z∈ Z.

Note that two weakly increasing functions need not be non decreasing

Contribution of various researchers in ordered metric space:

1.Ishak Altun and Hakan Simsek [11] :

To prove the main results in ordered metric space, authors introduced implicit relation with following three conditions:

S_1 : $S(u_1,u_2,u3,\ldots,u_6)$ is decreasing in variables $u_1,u_2,u3,\ldots,u_6$

S_2 : ∃aright continuous function g defined on set of positive reals where g(0)=0 and g(u) < u for u > 0 such that for v ≥ 0

S(v,w,v,w,0,v+w) ≤ 0

Or

S(v,w,0,0,w,w)≤ 0

This implies v ≤ g(w)

S_3 : S(v,0,v,0,0,v) > 0 for every v > 0

The authors proved a lemma in which they proved that $lt_{n\to\infty} g(u)=0$ when g^n=gogog….og (n-times) and g(u) < u for u > 0

The main result of authors is :

Theorem1.1 : Consider a poset (Y ,≼). Let d is a distance function on Y and (Y,d) is a metric space such that every cauchysequence in Y converges to some point in Y. Let G: Y→Y is an increasing function suchthat for every y,z ∈ Y with zis partially less or equal to y and

U(d(Gy,Gz),d(y,z),d(y,Gy),d(z,Gz),d(y,Gz),d(z,Gy)) ≤ 0

Where U ∈ζ

then G is continous function.

Or if $\{y_n\}$ proper subset of Y is a increasing sequence with y_n approaches to y in Y and

Y_n partially less or equal to y for all n holds if ∃y_0 ∈ Y such thaty_0 partially less or equal to $G(y_0)$.Then G has a fixed point.

Corollary 1.2 :Consider a poset (Y ,≼). Let d is a distance function on Y and (Y, d) is a metric space such that every cauchy sequence in Y converges to some point in Y . Let G:

Y→Y is an increasing function such that for every y,z ∈ Y with z partially less or equal to y and

$D(Gy,Gz) \leq \beta \max\{d(y,z),d(y,Gy),d(z,Gz)\}+(1-\beta)[cd(y,Gz)+ed(z,Gy)$

Where $0\leq\beta\leq1$, $0\leq c\leq½$, $0\leq e\leq1/2$. Then G is continuous function

Or if $\{y_n\}$ proper subset of Y is a increasing sequence with y_n approaches to y in Y and

$y_n$partially less or equal to y for all n holds if $\exists y_0 \in Y$ with y_0 partially less or equal to $G(y_0)$.Then G has a fixed point.

Theorem 1.3: Consider a poset (Y, ⪯). Let there is a distance function d on X and (Y,d) is metric space such that every cauchy sequence in Y converges to some point in Y.. Suppose H, I : Y → Y are two functions suchthat Hy is partially less or equal toIHy and Iy is partially less or equal to HIy and for every y,z □ Ywhere x ≤ y or y ≤ x

$V(d(Hy, Iz), d(y, z), d(y, Hy), d(z, Iz), d(y, Iz), d(z, Iz)) \leq 0$,

where $V \in\zeta$. Then

H is continous function.

or

I is continuous function.

or

if $\{y_n\}$ is proper subset of y is a non decreasing sequence with y_n approaching to y in Y

and y_n partially less or equal toy for every n hold

then H and I have a common fixed point

Corollary 1.4: Consider a poset (Y, ⪯) .Consider a complete metric space(X,d) on X. Consider two functions H,I :Y→Y such that for every y,z ∈ X where x ≤ y or y ≤ x

$d(Hy, Iz) \leq g(d(y, z))$,

where g is a function defined on positive realswith right hand limit and functional values of domain as same such that $g(0) = 0$, $g(u) < u$ for tu> 0.

Then

H is continuous function

or I is continuous function

or

if $\{Y_n\}$ proper subset of Y is a increasing sequence with y_n approaching to y in Y,

then y_n partially less or equal toy for all n hold, then H and I have a common fixed point.

2.Poonam Kumam et al [12]:

In order to prove main results authors introduced non linear contraction condition specified by a rational expression.

The main results of authors is :

Theorem 2.1. Consider adistance function d on set Y such that every cauchy sequence in Y converges to some point in Yand (Y, ⪯) is a poset provided with metric d on it. Consider two functions P, Q : Y → Y satisfying (for pairs (y, z) □ Y × Ysuch that Py ≤ Qz or Qz ≤ Py)the condition :

$l(Qy, Qz) \leq \gamma\{l(Py, Qy), l(Pz, Qz)/1+l(Py, Pz)\}1+\delta(Py,z)$,

where γ andδare positive reals with γ+δ is less than 1.

If the and the functions P ,Q and set Y preserve following two points :

(i)ForY∃ an increasing sequence $\{u_n\}$w.r.t relation⪯, in such a manner that $u_n$approaches u as n tend to∞ and u belongs to Y.This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers. .Functions P andQaresuch that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to $Q^{-1}(Py)$

(ii) the pair (P, Q) of self mappings is such that P*Q= Q*P and if ∃$\{u_n\}$

in Y such that QPu_napproaches Qu as ntends to ∞ and agrees with

$lt_{n\to\infty}Qu_n = lt_{n\to\infty}Pu_n = u$ where u belongs to Y

Then the functionsP and Q will have a point in their common domain with same image i.e., ∃v □ Y such that Pv = Qv.

Corollary 2..2. Consider a poset (Y, ⪯) Let dbe a distance function on non empty set Y such that in ordered pair(Y,d) every Cauchy sequence converges to some point in . SupposeP, Q :Y→ Yare twofunctions satisfying (for pairs (y, z) □ Y×Y such that Py ≤ Qz or Qz ≤ Py) the condition :

$l(Qy, Qz) \leq \gamma l(Py, Pz)$,

where γ is a positive real number such thatβ is less than 1.

If the functions P ,Q and set Y preserve following two points :

(i)For Y ∃ an increasing sequence $\{u_n\}$w.r.t relation ⪯, in such a manner that $u_n$approaches u as n tends to ∞ and u belongs to Y. This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers. Functions P and Q are such that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to $Q^{-1}(Py)$

(ii) The pair (P, Q) of self mappings is such that P*Q= Q*P and if ∃$\{u_n\}$

in Y such that QPu_n approaches Qu as n tends to ∞ and agrees with

$lt_{n\to\infty}Qu_n = lt_{n\to\infty}Pu_n = u$ where u belongs to Y

Then the functions P and Q will have a point in their common domain with same image i.e., ∃v □ Y such that Pv = Qv.

Theorem 2.3. Let d be a poset (Y,⪯).Let Y be any non empty set suchthatin the ordered pair(Y, d) every cauchy sequence converges to some point in Y. Consider two self functions P: X → X and Q:X → X satisfying (pairs (y,z) □ Y× Ysuchthat Py≤Qz or Qz ≤ Py)the condition :

(Qy, Qz) ≤ γ{ (Py, Qy).l(Pz, Qz)/1+(Py,Pz)}+ δ (Rx, Ry)

where γ, δare positive realsandγ+δis less than 1.

If thefunctions P ,Q and set Y preserve following two points :

(i)Functions P and Q are such that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to Q^{-1}(Py)

(ii)For the pair (P, Q) of self mappings ∃{u_n}in Y such that distance between Q$Pu_n$and PQu_napproaches zero as n approaches ∞.AlsoQPu_napproaches to Qu as n tends to ∞ and agrees with

$lt_{n\to\infty}Qu_n= lt_{n\to\infty}Pu_n$= u where u belongs to Y

Then the mfunctiond P and Q will have a point in their common domain with same image i.e., ∃v □ Y such that Pv = Qv.

Then the functions Q and P have a coincidence point. i.e.,∃w □ Y such that Pw = Qw.

Theorem 2.4. Consider a poset (Y, ⪯) with a distance function$d$ on Y.Let in ordered pair (Y, d) everyCauchy sequence converges to some point in Y where Y is some non empty set. Consider two self functions P : Y → Y and Q : Y → Y satisfying (for pairs (y, z) □ Y × Y such that Py ≤ Qz or Qz ≤ Py) the condition :

(Qy, TQz) ≤γ{ (y,Qz) · (z, Qz)/1 + l (y, z) }+ δ (x, y)

where γ, δ are positive reals such that γ+δis less than 1.

If the functionss P ,Q and set Y preserve following two points :

(i)Qy is partially less or equal to Q(Qy)∀y □ Y,

(ii)FunctionsQ should be such that:

(a)it should be defined on each point of its domain

(b) its limit should exist

(c) functional value should be equal to limiting value

(iii) For Y ∃ an increasing sequence {u_n}w.r.t relation ⪯, in such a manner that $u_n$approaches u as n tends ∞ and u belongs to Y. This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers is regular.

Then, ∃ u in Y suchtat Qu=u.

Theorem 2.5. Consider a poset (Y, ⪯)with adistance function$d$ on non empty set Y. Let in ordered pair(Y, d) every Cauchy sequence converges to some point in Y. Consider two self functions P : Y → Y and Q : Y → Y satisfying (for pairs (y, z) □ Y × Ysuch that Py ≤ Qz or Qz ≤ Py) the condition:

l(Qy, Qz) ≤ γ{l (Py, Qy) ·(Pz, Qz)/1+l(Py, Pz)} +δl (Py, Pz)

where γ, δare positive reals and γ+ δis less than 1.

If the functions P ,Q and set Y preserve following three points :

(i)For Y ∃ an increasing sequence $\{u_n\}$w.r.t relation ⪯, in such a manner that $u_n$approaches u as n tends to ∞ and u belongs to Y. This will imply that u_n is partially less or equal to u for all n belonging to set of natural numbers. Functions P and Q are such that for every y belonging to Y, Py is partially less or equal to Pz for all z belonging to $Q^{-1}(Py)$

(ii) the pair (P, Q) of self mappings is such that P*Q= Q*P and if ∃$\{u_n\}$

in Y such that QPu_n approaches Qu as n tends to ∞ and agrees with

$lt_{n\to\infty}Qu_n = lt_{n\to\infty}Pu_n = u$ where u belongs to Y

(iii) Pis such that for a increasing sequence $P(u_n)$ inY if $P(u_n)$ approaches u as n tends to ∞ then $P(u_n)$ is less or equal to P(u) or P(u) is less or equal to $P(u_n)$ ∀ n n belonging to set of naturals.

Then the functions P and Q have a common fixed point.

3. Uniqueness results in [12]

In this section we examine therequirement under which Theorem 2.1 assuresthe uniqueness of common fixed point.

Theorem 3.1. Besides the assumptions of Theorem 2.1,let every (y, y∗) □Y × Y, ∃bw□ Y such that Qw is upper bound of Qy*and*Qy∗, then for the functionsP and Q∃ u and v in Ysuch thatPu= u and Qv = v implies u=Pu=Qv= v..

Theorem 3.2. :Besides the assumptionsof Theorem 2.4., suppose for pairs(y, y∗) ∈Y×Y∃ w∈ Y such that Qw is upper bound of Qy *and* Qy*,then for the functions P and Q ∃ u and v in Y such that Pu= u and Qv = v implies u=Pu=Qv= v..

Corollary 3.3. : Besides the assumptions of Theorem 2.1. (or Theorem 2.2.),if that for every pair (y, y∗)□Y×Y there exists bw□Y such that Qw is upper bound of Qy and Qy∗.Then∃a point y belonging to Y such that y =Qy and this y is unique.

4.A.R. Butt et al [13]:

To prove the main results on ordered metric space introduced implicit relation withthre conditions which are

H_1 : $H(a_1,\ldots,a_6)$ is decreasing in variables a_5,a_6

H :∃k□ [0,1) such that

$$G(a,b,b,a,a+b,0) \leq 0$$

Or

$$G(a,b,a,b,0,a+b) \leq 0$$

This implies

$$a \leq kb$$

H_3 : $H(a,0,0,a,a,0) > 0$, for everya> 0..

The main results of the authors is :

To prove the main results he authors have considered a metric space (A,d) such that every cauchy sequence in A converges to some point in A, where set A is having partial order andordered pair (A, ⪯) is a partially ordered set.

The consideration of contractive condition by authors is as:

$$H(l(mu, mv),l(u,v),d(u,mu),l(v,mv),l(u,mv),l(v,mu)) \leq 0. \qquad (B)$$

Theorem 4.1.: Consider a self mapping m on set A be such that ∃ a number k which belongs to semi left closed right open interval 0 and 1 with the contractive condition (1-k)l(u,mu)≤l(u,v) which gives the condition (B) for all elements u,v of A such that either u≤ v or v≤u and for some H□ a.If the set A and function m preserve the following points:

i)∃u_0∈A such that u_0 ≤ m

ii)if u,v∈A is such that u≤ v then mu≤ m

iii)if u_n is any sequence in A and u_n approaches u and for all terms of{u_n} such thateither u ≤ v or v ≤ u where u and v are any two terms of {u_n }, then u_n is partially less or equal to u∀ n.

Then ∃ a point u in set A such that u if fixed point of m.

Theorem 4.2.: Consider a self mapping m on set A be such that ∃ a number k which belongs to semi left closed right open interval 0 and 1 with the contractive condition (1-k)l(u,mu) ≤ l(u,v) which gives the condition (B) for all elements u,v of A such that either u ≤ v or v ≤ u and for some H □ a. If the set A and function m preserve the following points:

i)∃ u_0∈A such that mu_0 ≤ u_0

ii)if u,v∈A is such that u≤ v then mv≤ m

iii)if u_n is any sequence in A andu_n approaches u and for all terms of{u_n} such that either u ≤ v or v ≤ u where u and v are any two terms of {u_n }, then u_n is partially less or equal to u ∀ n.

Then ∃ a point u in set A such that u if fixed point of m

Theorem 4.3.:Consider a self mapping m on set A be such that ∃ a number k which belongs to semi left closed right open interval 0 and 1 with the contractive condition (1-k)l(u,mu) ≤ l(u,v) which gives the condition (B) for all elements u,v of A such that either u ≤ v or v ≤ u and for some H □ a.
Let function m is either increasing or decreasing. If the set A and function m preserve the following points:

i)∃u_0∈such that mu_0 ≤u_0

ii)if u_n is any sequence in A andu_n approaches u and for all terms of{u_n} such that
either u ≤ v or v ≤ u where u and v are any two terms of {u_n }, then u_n is partially less or equal to u ∀ n

Then ∃ a point u in set A such that u if fixed point of m

Conclusion :

In this paper some fixed point results are given which are quite simple with straight forward proof. The distinctive feature of this paper is that various developments in fixed point theory in the field of ordered metric space has been presented step by step. The significance of this Paper is :

i)Existence theorem for common solution of two integral equations has been presented.

ii)Fixed Point results using non linear contraction condition and existence theorem for solution of an integral equation has been presented.

iii)Fixed point theorems for uniqueness of fixed point using non linear rational contraction has been presented.

iv)Fixed point results using specifiedimplicit relation has been presented which are applicable to homotopy.

References

[1]S. K. Chatterjea, Fixed point theorem, C. R. Acad. Bulgare Sci., 25(1972), 727 730.

[2]T. GnanaBhaskar and V. Lakshmikantham," Fixed point theorems in partially ordered metric spaces and applications", *Nonlinear Anal.*, 65(2006), 1379 -1393.

[3]J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets", *Nonlinear Anal.*, 71(2009), 3403-3410.

[4]W. A. Kirk and K. Goebel, "Topics in Metric Fixed Point Theory", Cambridge University Press, Cambridge 1990.

[5]A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[6] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.

[7] J. J. Nieto and R. Rodr´ıguez-L´opez, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," Order, vol. 22, no. 3, pp. 223–239, 2005.

[8] J. J. Nieto and R. Rodr´ıguez-L´opez, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[9] J. J. Nieto, R. L. Pouso, and R. Rodr´ıguez-L´opez, "Fixed point theorems inordered abstract spaces," *Proceedings of the American Mathematical Society*, vol. 135, no. 8, pp. 2505–2517, 2007.

[10]D. O'Regan and A. Petrus¸el, "Fixed point theorems for generalized contractionsin ordered metric spaces," *Journal of Mathematical Analysis and Applications,* vol. 341, no. 2, pp. 1241–1252, 2008.

[11] I. Altun and H. Simsek, "Some fixed fixed point theorems on ordered metric spaces and applications, *Fixed Point Theory Appl.*, 2010(2010), 17 pages.

[12]P. Kumam, F.Rouzkard, M. Imdad and D. Gopal,,*"*Fixed point theorems on ordered metric spaces through a rational contraction*"* ,*Abstract and applied analysis*, vol. 2013, 8 pages ,2013.

[13]A. R. Butt,I. Beg, A. Iftikar," Fixed points on ordered metric spaces with application in homotopy theory," *Journal of fixed point and applications,* vol. 20, no.2, 15 pages, 2018.

[14]Babach, S. Surles, Operations dabseles ensembles abstracts at leur applicationsand equations integrals, Fund. Math Sci. 3(1992). © 2014, IJSRMSS All Rights Reserved 46

[15]B.C.Dhage, "Condensing mappings and applications to existence theorems for common solution of differential equations," *Bulletin of the Korean Mathematical Society,* vol. 36, no. 3, pp. 565–578, 1999.

[16]B. C. Dhage, D. O'Regan, and R. P. Agarwal, "Common fixed point theorems for a pair of countably condensing mappings in ordered Banach spaces," *Journal of Applied Mathematics and Stochastic Analysis,* vol. 16, no. 3, pp. 243–248, 2003.