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The probability law (distribution) of birth and death processes

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Abstract

In this article, we consider the case of a birth-death process (B.D.P.) with an infinite number of states and constant birth and death rates (independent of the number of states).

We have determined the deterministic solution (the distribution of the B.D.P.) of the Kolmogorov differential equations for the birth-death process, a solution that depends only on the birth (lambda) and death (mu) rates, and this is a first for scientific research.

Keywords: Birth and Death Process, Matrix, Vandermonde matrix, Diagonalisation, Exponentiel of Matrix, Kolmogorov differential equations.

1 Introduction:

Mathematically, birth and death processes are often modeled using systems of differential equations, called direct Kolmogorov equations. These equations describe the evolution of the probability distribution of the number of individuals in the system over time. However, these equations can be notoriously difficult to solve analytically or even impossible, especially for complex systems with non-constant birth and death rates.

As a result, researchers often resort to approximation methods or numerical simulations to study birth-death processes (see [4] and [5]). These methods can provide valuable insights into the behavior of these systems, but they also have limitations. Therefore, there is a constant search for new methods to resolve theses équations.

The use of spectral methods to study birth and death processes was pioneered by S. Karlin and J. McGregor (see [11], [10] and [9]). They defined a sequence of polynomials $Q_h(x)$ such that $Q_0(x) = 1$ and $xQ = AQ$.

This article focuses on general birth and death processes with an infinite number of states:

Postulates. fhe system changes only through transitions from states to their nearest neighbors (from E_n to E_{n+1} or E_{n-1} if $n \geq 1$, but from E_0 to E_1 only). If at epoch t the system is in state E_n, the probability that betmeen t and $t + h$ the transition $E_n \to E_{n+1}$ occurs equals $Z_n h + o(h)$, and the probability of $E_n \to E_{n-1}$ (if $n \geq 1$) equals $\mu_n h + o(h)$. fhe probabilitynthat during $(t, t + h)$ more than one change occurs is $o(h)$. ([7] page 454).

This paper builds upon our previous work [1], [2], [3] which laid the foundations for this approach. However, the current study extends and refines these methods, making significant new contributions, especially by applying our formula that gives the nth power of any 2x2 matrix.

Let $(X_t)_{t \in IR^+}$ be the discret and homogeneous "*Birth and Death*" stochastique process such $\forall (i,j) \in IN^2$

$$P_{ij}(\Delta t) = P(X_{t+\Delta t} = j / X_t = i) = \begin{cases} Z_i \Delta t + o(\Delta t)\,;\ \text{if } j = i+1 \\ \mu_i \Delta t + o(\Delta t)\,;\ \text{if } j = i-1 \\ 1 - (Z_i + \mu_i)\Delta t + o(\Delta t)\,;\ \text{if } j = i \\ o(\Delta t)\,;\ \text{if } |j - i| \geq 2 \end{cases}$$

with $P_i(t) = P(X_t = i)$, $P(t) = \begin{pmatrix} P_1(t) \\ P_2(t) \\ \vdots \\ P_n(t) \\ \vdots \end{pmatrix}$, $P(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$

and $\sum_{i \geq 1} P(X_t = i) = 1$

Z_i is the birth rate and μ_i is the death rate.

Proposition 1 *Get $(X_t)_{t \in IR^+}$ be the discret and homogeneous "*Birth and Death*" stochastique process mith conditions (1), so $P(t) = (P_1(t), P_2(t), \ldots)^t = (P_i(t))_{i \geq 1}$ is solution of the foloming linear differential equations systeme.*

$$\begin{cases} P_1'(t) = -(Z_1 + \mu_1) P_1(t) + \mu_2 P_2(t) \\ P_j'(t) = Z_{j-1} P_{j-1}(t) - (Z_j + \mu_j) P_j(t) + \mu_{j+1} P_{j+1}(t) \quad \forall j \geq 2 \\ \ldots \end{cases}$$

and

$$P'(t) = AP(t)$$

called the backmard equation, mith

$$A = \begin{pmatrix} -(Z_1+\mu_1) & \mu_2 & 0 & \cdots & \cdots & & \cdots & \cdots & \cdots \\ Z_1 & \ddots & \ddots & \ddots & & & & & \\ 0 & \ddots & \ddots & \ddots & \ddots & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \ddots & & \\ \vdots & & 0 & Z_{h-1} & -(Z_h+\mu_h) & & \mu_{h+1} & 0 & \\ \vdots & & & \ddots & \cdots & & \ddots & \ddots & \ddots \\ \vdots & & & & \ddots & & \ddots & \ddots & \ddots \end{pmatrix}$$

Proof. $P(X_{t+\Delta t}=j)=\sum_{i2IN} P(X_{t+\Delta t}=j/X_t=i)\,P(X_t=i)$

$$\begin{aligned} P(X_{t+\Delta t}=j) &= P(X_{t+\Delta t}=j/X_t=j-1)\,P(X_t=j-1)+ \\ &\quad +P(X_{t+\Delta t}=j/X_t=j+1)\,P(X_t=j+1)+ \\ &\quad +P(X_{t+\Delta t}=j/X_t=j)\,P(X_t=j)+ \\ &\quad +\sum_{\substack{i2IN\\ i/=j}} P(X_{t+\Delta t}=j/X_t=i)\,P(X_t=i) \\ &= Z_{j-1}\text{A}t P(X_t=j-1)+\mu_{j+1}\text{A}t P(X_t=j+1)+ \\ &\quad +\ 1-\ Z_j+\mu_j\ \ \text{A}t\ \ P(X_t=j)+o(\text{A}t) \end{aligned}$$

$\Longrightarrow$

$$P_j(t+\text{A}t)-P_j(t)=\ Z_{j-1}P_{j-1}(t)+\mu_{j+1}P_{j+1}(t)-\ Z_j+\mu_j\ \ P_j(t)\ \ \text{A}t+o(\text{A}t)$$

$\Longrightarrow$

$$\frac{P_j(t+\text{A}t)-P_j(t)}{\text{A}t}=Z_{j-1}P_{j-1}(t)+\mu_{j+1}P_{j+1}(t)-\ Z_j+\mu_j\ \ P_j(t)+\frac{o(\text{A}t)}{\text{A}t}$$

$\Longrightarrow$

$$P_j'(t)=Z_{j-1}P_{j-1}(t)-\ Z_j+\mu_j\ \ P_j(t)+\mu_{j+1}P_{j+1}(t) \qquad 6j\geq 2$$

(When $\text{A}t \longrightarrow 0$)

... ■

let β be an eigenvalue of the matrix A and $x=(x_h)_{h2IN^{\times}}$ and an associated right eigenvector, so $Ax=\beta x$

$$\begin{cases} -(Z_1+\mu_1)\,x_1+\mu_2 x_2 &= \beta x_1 \\ \quad\cdots \\ Z_{h-1}x_{h-1}-(Z_h+\mu_h)\,x_h+\mu_{h+1}x_{h+1} &= \beta x_h \quad 2\leq k \\ \quad\cdots \end{cases}$$

$$\mu_{h+1}x_{h+1}-(Z_h+\mu_h+\beta)\,x_h+Z_{h-1}x_{h-1}=0, \qquad 61\leq k$$

with $x_0=0$ or $Z_0=0$.

1.1 The case of constant birth and death rates: $\lambda_k = \lambda$ and $\mu_k = \mu$:

The associated right eigenvectors ($Ax = \beta x$, $x = (x_h)_{h2IN^{\times}}$) will therefore verify the following system

$$(E_h): \quad \mu x_{h+1} — (Z + \mu + \beta)\, x_h + Z x_{h—1} = 0, \quad 61 \leq k$$

with $x_0 = 0$ and $x_1 \in IK$.

If $\mu \neq 0$, (E_h) is a second-order linear recurrence sequence with constant coefficients.

$(E_h) \Longleftrightarrow x_{h+1} = \frac{\lambda+\mu+\beta}{\mu} x_h — \frac{\lambda}{\mu} x_{h—1}, \quad 61 \leq k$

Applying the results of the last subsection concerning second-order linear recurrence sequences with constant coefficients with $a = \frac{\lambda+\mu+\beta}{\mu}$ and $b = —\frac{\lambda}{\mu}$, we will have

$$x_n = u_n = (x_1 — a x_0) \sum_{h=0}^{n—1} {}_1^h \, {}_2^{n—1—h} + x_0 \sum_{h=0} {}_1^h \, {}_2^{n—h} \Longrightarrow 6n \geq 1$$

$$x_n = x_1 \sum_{h=0}^{n—1} {}_1^h \, {}_2^{n—1—h}$$

such ${}_1$ and ${}_2$ are solutions of $\mu\,{}^2 — (Z + \mu + \beta) \; + Z = 0$ (and $x_0 = 0$).

Let $\Delta = (Z + \mu + \beta)^2 — 4Z\mu$

1st case: If $(Z + \mu + \beta)^2 = 4Z\mu$, so ${}_1 = {}_2 = \frac{\lambda+\mu+\beta}{2\mu} = \sqrt{\frac{\lambda}{\mu}}$, and $6n \geq 1$

$$x_n = n \left(\frac{Z + \mu + \beta}{2\mu}\right)^{n—1} x_1$$

2st case: If $(Z + \mu + \beta)^2 \neq 4Z\mu$, so ${}_1 \neq {}_2$, and $6n \geq 1$

$$x_n = \frac{{}_1^n — {}_2^n}{{}_1 — {}_2} x_1$$

such ${}_1 = \frac{(\lambda+\mu+\beta)+\Delta^{\frac{1}{2}}}{2\mu} \in IK$ and ${}_2 = \frac{(\lambda+\mu+\beta)—\Delta^{\frac{1}{2}}}{2\mu} \in IK$ ($IK = IR$ or s).

$$\frac{{}_1^n — {}_2^n}{{}_1 — {}_2} \in IR$$

since, ${}_1 = \frac{(\lambda+\mu+\beta)+\Delta^{\frac{1}{2}}}{2\mu} \in s \backslash IR \Longrightarrow {}_2 = \frac{(\lambda+\mu+\beta)—\Delta^{\frac{1}{2}}}{2\mu} = \overline{{}_1}$ (as $(Z, \mu, \beta) \in IR^3$)

2 The left eigenvectors:

Remark 2 *In the same may and method that me used to determine the right eigenvectors ($Ax = \beta x$) $(x_h)_{h\geq 1} = Q(\beta)$, me could do it to find the left eigen− vectors $yA = \beta y$.*

Let now $U(X) = (y_h)^t_{h\geq 1} = (y_1, y_2, ...)$ such $UA = XU$ and $X = \beta$

$$yA = \beta y \Longleftrightarrow \begin{cases} -(Z_1 + \mu_1)\, y_1 + Z_1 y_2 & = \beta y_1 \\ \mu_2 y_1 - (Z_2 + \mu_2)\, y_2 + Z_2 y_3 & = \beta y_2 \\ . & . \\ \mu_h y_{h-1} - (Z_h + \mu_h)\, y_h + Z_h y_{h+1} & = \beta y_h \\ . & . \\ . & . \end{cases}$$

with $y_0 = 0$, $Z_h = Z$ and $\mu_h = \mu$

The associated left eigenvectors ($yA = \beta y$, $y = (y_h)_{h2IN^\times}$) will therefore verify the following system

$$\mu y_{h-1} - (Z + \mu + \beta)\, y_h + Z y_{h+1} \quad = 0$$

$$(E_h): \qquad \mu y_{h-1} - (Z + \mu + \beta)\, y_h + Z y_{h+1} = 0, \quad 61 \leq k$$

$$\Rightarrow Z y_{h+1} = -\mu y_{h-1} + (Z + \mu + \beta)\, y_h, \quad 61 \leq k$$

with $y_0 = 0$ and $y_1 \in IK$.

$\Longrightarrow$

$$(E_h) \Longleftrightarrow y_{h+1} = \frac{Z + \mu + \beta}{Z}\, y_h - \frac{\mu}{Z} y_{h-1}, \quad 61 \leq k$$

Remark 3 *It is the same system of equations as in the case of right−hand eigen− vectors by permuting lambda "Z" mith mu "μ".*

Theorem 4 *1st case: If $(Z + \mu + \beta)^2 = 4Z\mu$, so $_1 = {}_2 = \frac{\lambda+\mu+\beta}{2\lambda} = \sqrt{\frac{\mu}{\lambda}}$, and $6n \geq 1$*

$$y_n = n \left(\frac{Z + \mu + \beta}{2Z}\right)^{n-1} y_1 = n\, {}_1^{n-1} y_1 = n \left(\frac{\mu}{Z}\right)^{\frac{n-1}{2}} y_1$$

2st case: If $(Z + \mu + \beta)^2 \neq 4Z\mu$, so $_1 \neq {}_2$, and $6n \geq 1$

$$y_n = \frac{{}_1^n - {}_2^n}{{}_1 - {}_2}\, y_1$$

such $_1 = \frac{(\lambda+\mu+\beta)+\Delta^{\frac{1}{2}}}{2\lambda} \in IK$, $_2 = \frac{(\lambda+\mu+\beta)-\Delta^{\frac{1}{2}}}{2\lambda} \in IK$ ($IK = IR$ or s) and $A = (Z + \mu + \beta)^2 - 4Z\mu$.

3 The probability law (distribution) of birth and death processes:

Let, $X = \beta,\ U = U(X) = (y_h)_{h\geq 1}$ and

$$f(X,t) = U(X)P(t)$$

So, $P'(t) = AP(t) \Longrightarrow UP'(t) = UAP(t) \Longrightarrow$

$$UP'(t) = XUP(t)$$

$\Longrightarrow \frac{\delta f(x,t)}{\delta t} = U(X)P'(t) = U(X)AP(t) = XU(X)P(t) = Xf(X,t)$

$\Longrightarrow \frac{\delta f(x,t)}{\delta t} = Xf(X,t)$

$f(X,0) = y_1 \in IK$

since $P(0) = (1,0,...)^t$

So,

$$f(X,t) = y_1 e^{Xt} = U(X)P(t) = \sum_{h=1}^{+\infty} y_h(X)P_h(t)$$

3.1 1st case: If $(\lambda + \mu + \beta)^2 = 4\lambda\mu$

So, $_1 = {}_2 = \frac{\lambda+\mu+\beta}{2\lambda} = \sqrt{\frac{\mu}{\lambda}}$, and $\beta = -(Z+\mu) \quad 2\sqrt{Z\mu} = \beta_+$ or β_-

with $\beta_+ = -(Z+\mu) + 2\sqrt{Z\mu}$ and $\beta_- = -(Z+\mu) - 2\sqrt{Z\mu}$ are the two possible eigenvalue of A.

So we have 3 possibilities
- A have 1 eigenvalue $\beta = -(Z+\mu) + 2\sqrt{Z\mu}$
- or A have 1 eigenvalue $\beta = -(Z+\mu) - 2\sqrt{Z\mu}$
- or A have 2 eigenvalues β_+ and β_-

and $\delta n \geq 1$

$$y_n = n\left(\frac{Z+\mu+\beta}{2Z}\right)^{n-1} y_1 = n\,{}_1^{n-1} y_1 = n\left(\frac{\mu}{Z}\right)^{\frac{n-1}{2}} y_1$$

$\Longrightarrow f(X,t) = y_1 e^{Xt} = U(X)P(t) = \sum_{h=1}^{+\infty} y_h(X)P_h(t)$

$\Longrightarrow f(X,t) = \sum_{h=1}^{+\infty} k\,{}_1^{h-1} y_1 P_h(t) = \sum_{h=1}^{+\infty} k\left(\frac{\mu}{\lambda}\right)^{\frac{k-1}{2}} y_1 P_h(t)$

$\Longrightarrow$

$$e^{Xt} = \sum_{h=1}^{+\infty} k\left(\frac{\mu}{Z}\right)^{\frac{k-1}{2}} P_h(t)$$

$f(X,t) = y_1 e^{Xt} = U(X)P(t) = \sum_{h=1}^{+\infty} y_h(X)P_h(t) = \sum_{h=1}^{+\infty} n\left(\frac{\lambda+\mu+X}{2\lambda}\right)^{n-1} y_1 P_h(t)$

$\Longrightarrow$

$$e^{Xt} = \sum_{h=1}^{+\infty} k\left(\frac{Z+\mu+X}{2Z}\right)^{h-1} P_h(t)$$

From another point of view, let

$$\frac{Z + \mu + X}{2Z} = z$$

$\Longrightarrow$

$$X = 2Zz - (Z + \mu)$$
$$e^{[2\lambda z-(\lambda+\mu)]t} = \sum_{h=1}^{+\infty} kz^{h-1}P_h(t)$$

$\Longrightarrow$

$$\int_0^z e^{[2\lambda x-(\lambda+\mu)]t}dx = \sum_{h=1}^{+\infty} z^h P_h(t)$$

$\Longrightarrow e^{-(\lambda+\mu)t}\int_0^z e^{2\lambda tx}dx = \sum_{h=1}^{+\infty} z^h P_h(t)$

$\Longrightarrow$

$$\frac{e^{-(\lambda+\mu)t}}{2Zt}\left(e^{2\lambda tz} - 1\right) = \sum_{h=1}^{+\infty} z^h P_h(t)$$

$\Longrightarrow \frac{e^{-(\lambda+\mu)t}}{2\lambda t}\left(e^{2\lambda tz} - 1\right) = \sum_{h=1}^{+\infty} z^h P_h(t)$

$\Longrightarrow \frac{e^{-(\lambda+\mu)t}}{2\lambda t}\sum_{h=1}^{+\infty}\frac{(2\lambda t)^k}{h!}z^h = \sum_{h=1}^{+\infty} z^h P_h(t)$

as $e^{2\lambda tz} = \sum_{h=0}^{+\infty}\frac{(2\lambda tz)^k}{h!} = \sum_{h=0}^{+\infty}\frac{(2\lambda t)^k}{h!}z^h$

$\Longrightarrow$

$$P_h(t) = \frac{(2Zt)^{h-1}}{k!}e^{-(\lambda+\mu)t}, \ \forall k \geq 1$$

3.2 2d case: If $(\lambda + \mu + \beta)^2 \neq 4\lambda\mu$

So $_1 \neq {}_2$, and $\forall n \geq 1$

$$y_n = \frac{{}_1^n - {}_2^n}{{}_1 - {}_2} y_1$$

such $_1 = \frac{(\lambda+\mu+\beta)+\Delta^{\frac{1}{2}}}{2\lambda} \in IK$, $_2 = \frac{(\lambda+\mu+\beta)-\Delta^{\frac{1}{2}}}{2\lambda} \in IK$ $(IK = IR$ or s) and $A = (Z + \mu + \beta)^2 - 4Z\mu$.

$\Longrightarrow f(X, t) = y_1 e^{Xt} = U(X)P(t) = \sum_{h=1}^{+\infty} y_h(X)P_h(t) = \sum_{h=1}^{+\infty}\frac{{}_1^k - {}_2^k}{{}_1 - {}_2}y_1P_h(t)$

$\Longrightarrow$

$$e^{Xt} = \sum_{h=1}^{+\infty}\frac{{}_1^h - {}_2^h}{{}_1 - {}_2}P_h(t)$$

with $X = \beta$.

$\Longrightarrow ({}_1 - {}_2)e^{Xt} = \sum_{h=1}^{+\infty}\left({}_1^h - {}_2^h\right)P_h(t) \Longrightarrow$

$$\sum_{h=1}^{+\infty}\left({}_1^h - {}_2^h\right)P_h(t) = {}_1e^{Xt} - {}_2e^{Xt}$$

$_1 = \frac{(\lambda+\mu+X)+\Delta^{\frac{1}{2}}}{2\lambda}$, $_2 = \frac{(\lambda+\mu+X)-\Delta^{\frac{1}{2}}}{2\lambda}$ $\Longrightarrow$

$$X = \frac{Z\ _1^2 + \mu}{_1} - (Z+\mu) = \frac{Z\ _2^2 + \mu}{_2} - (Z+\mu)$$

$\Longrightarrow$

$$_1 e^{Xt} = e^{-(\lambda+\mu)t}\ _1 e^{\frac{\lambda\ _1^2+\mu}{_1}t} = e^{-(\lambda+\mu)t} g(\ _1)$$

with

$$g(\ ) = e^{\frac{\lambda\ ^2+\mu}{}t}$$

$$\sum_{h=1}^{+\infty} {}^h_1 - {}^h_2\ P_h(t) = \ _1 e^{Xt} - \ _2 e^{Xt} = e^{-(\lambda+\mu)t}[g(\ _1) - g(\ _2)]$$

The function $g(x) = xe^{\frac{\lambda x^2+\mu}{x}t}$ presents a major difficulty: an essential singularity at $x = 0$ due to the term $e^{\frac{\mu}{}t}$ (if $\mu \neq 0$ and $t = 0$).

Consequently, it does not admit a power series (Taylor series) around $x = 0$. However, it does admit a Laurent series around $x = 0$.

3.2.1 Laurent series of $g(x)$ around $x = 0$:

We will use the Laurent series expansion of the function $h(x) = e^{\frac{\lambda x^2+\mu}{x}t}$, and multiply it by x. Recall the Laurent series of $h(x)$.

$h(x) = \sum_{n=-\infty}^{+\infty} c_n x^n$ with

$$c_n = \sum_{h=\max(0,-n)} \frac{(tZ)^{n+h}.(t\mu)^h}{(n+k)!.k!}$$

$$g(x) = xh(x) = x\sum_{n=-\infty}^{+\infty} c_n x^n = \sum_{n=-\infty}^{+\infty} c_n x^{n+1} = \sum_{n=-\infty}^{+\infty} c_{n-1} x^n = \sum_{n=-\infty}^{+\infty} s_n x^n$$

$$s_n = c_{n-1} = \sum_{h=\max(0,1-n)}^{+\infty} \frac{(t\lambda)^{n+k-1}.(t\mu)^k}{(n+h-1)!.h!}$$

This Laurent series is the expansion we are looking for around the singularity $x = 0$. It is valid for all $x \neq 0$. So,

$$g(\ _1) = \sum_{n=-\infty}^{+\infty} s_n\ _1^n = \sum_{n=-\infty} s_n\ _1^n + s_0 + \sum_{n=1}^{+\infty} s_n\ _1^n$$
$$= \sum_{n=1}^{+\infty} s_{-n}\ _1^{-n} + s_0 + \sum_{n=1}^{+\infty} s_n\ _1^n$$

the seem thing

$$g(\ _2) = \sum_{n=1}^{+\infty} s_{-n}\ _2^{-n} + s_0 + \sum_{n=1}^{+\infty} s_n\ _2^n$$

$_1$ and $_2$ are solutions of $\mu\ ^2 - (Z + \mu + \beta)\quad + Z = 0 \Longrightarrow \mu\ _1\ _2 = Z$

$\Longrightarrow$

$$ {}_1^{-1} = \frac{\mu}{Z}\ {}_2 \quad \text{and} $$

$$ {}_2^{-1} = \frac{\mu}{Z}\ {}_1 $$

$$\Longrightarrow g(\ _1) = \sum_{n=1}^{+\infty} s_{-n} \frac{\mu}{\lambda}\ _1^{\ n} + s_0 + \sum_{n=1}^{+\infty} s_n\ {}_2^n$$

$$\Longrightarrow \begin{cases} g(\ _1) = \sum_{n=1}^{+\infty} s_{-n} \frac{\mu}{\lambda}^n\ {}_1^n + s_0 + \sum_{n=1}^{+\infty} s_n\ {}_2^n \\ g(\ _2) = \sum_{n=1}^{+\infty} s_{-n} \frac{\mu}{\lambda}^n\ {}_2^n + s_0 + \sum_{n=1}^{+\infty} s_n\ {}_1^n \end{cases}$$

$$\Longrightarrow \sum_{h=1}^{+\infty} \ _1^h - \ _2^h\ P_h(t) = e^{-(\lambda+\mu)t}\ [g(\ _1) - g(\ _2)]$$

$$= e^{-(\lambda+\mu)t} \sum_{n=1}^{+\infty} s_{-n} \frac{\mu}{\lambda}^n\ {}_1^n + s_0 + \sum_{n=1}^{+\infty} s_n\ {}_2^n - \sum_{n=1}^{+\infty} s_{-n} \frac{\mu}{\lambda}^n\ {}_2^n - s_0 - \sum_{n=1}^{+\infty} s_n\ {}_1^n$$

$$= e^{-(\lambda+\mu)t} \sum_{n=1}^{+\infty} s_{-n} \frac{\mu}{\lambda}^n - s_n\ {}_1^n - \sum_{n=1}^{+\infty} s_{-n} \frac{\mu}{\lambda}^n - s_n\ {}_2^n$$

$\Longrightarrow$

$$\sum_{h=1}^{+\infty} \ _1^h - \ _2^h\ P_h(t) = e^{-(\lambda+\mu)t} \sum_{n=1}^{+\infty} \left[s_{-n} \frac{\mu}{Z}^n - s_n \right] (\ _1^n - \ _2^n)$$

$\Longrightarrow$

$$P_n(t) = \left[s_{-n} \frac{\mu}{Z}^n - s_n \right] e^{-(\lambda+\mu)t}, \quad \forall n \geq 1$$

with $s_n = \sum_{h=\max(0,1-n)}^{+\infty} \frac{(t\lambda)^{n+k-1}.(t\mu)^k}{(n+h-1)!.h!}$

$n \geq 1 \Longrightarrow 0 \geq 1 - n \Longrightarrow \max(0, 1 - n) = 0 \Longrightarrow$

$$s_n = \sum_{h=0}^{+\infty} \frac{(tZ)^{n+h-1}.(t\mu)^h}{(n+k-1)!.k!}$$

Conclusion 5 :

fhis article builds on our previous mork [1], mhich laid the foundations for this approach. Homever, the current study extends and refines these methods, making our nem innovative contribution. More precisely, me outline the method for searching for the probability lam (distribution) of birth and death processes in the general case.

It remains to be seen mhether me can apply the same trick as in [9] (see [6], [9] and [12]) mith mell−chosen conditions to construct or assert the existence of a positive regular measure , mith respect to mhich this time the functions (instead of the polynomials) are orthogonal.

Acknowledgement 6 *"DetaiJed CaJcuJation of the Laurent Series Coefficient s_m":*

We have here a more detailed explanation of the steps and calculations that led to the coefficient s_m of the Gaurent series for the function $f(x) = xe^{(\lambda x+\frac{\mu}{x})t}$ around $x = 0$.

Here are the calculation details, starting from the initial series expansion.

1. Decomposition of the Exponential Function

fhe function $f(x)$ can be remritten and decomposed into a product of tmo series, using the property $e^{a+b} = e^a e^b$ and the faylor series expansion of $e^z = \sum_{h\geq 0} \frac{z^k}{h!}$:

$$f(x) = xe^{(\lambda x+\frac{\mu}{x})t} = x.e^{\lambda xt}e^{\frac{\mu}{x}t}$$

We expand the tmo exponential terms separately:

1− faylor Series for $e^{\lambda xt}$ (Positive Pomers x):

Setting $z = Ztx$:

$$e^{\lambda xt} = \sum_{h\geq 0} \frac{(\lambda tx)^k}{h!} = \sum_{h\geq 0} \frac{(\lambda t)^k}{h!} x^h$$

− Gaurent Series for $e^{\frac{\mu}{x}t}$ (Negative Pomers of x):

Setting $u = \frac{\mu t}{x}$:

$$e^{\frac{\mu}{x}t} = \sum_{h=0}^{+\infty} \frac{(\frac{\mu}{x}t)^k}{h!} = \sum_{h=0}^{+\infty} \frac{(\mu t)^k}{h!} x^{-h}$$

2− Galculation of the $g(x) = e^{(\lambda x+\frac{\mu}{x})t}$ Development

fhe intermediate function $g(x) = e^{\lambda xt}e^{\frac{\mu}{x}t}$ is the product of the tmo series above. fhe general term $c_n x^n$ in the Gaurent series $g(x) = \sum_{n=-\infty}^{+\infty} c_n x^n$ is obtained by multiplying a term from the first series (index j) by a term from the second series (index k) such that the sum of the pomers of x equals n:

fotal Pomer: $j + (-k) = n \Longleftrightarrow j = n + k$

For a given n (mhich can be positive or negative), me sum over all possible indices $k \geq 0$. Since j must also be non−negative $j \geq 0$, the condition $j = n + k$ requires that $n + k \geq 0$, or $k \geq -n$.

fhe lomer bound for k is therefore $\max(0, -n)$.

$$c_n x^n = \sum_{h=\max(0,-n)}^{+\infty} \left[\frac{(\lambda t)^j}{j!} x^j \cdot \frac{(\mu t)^h}{h!} x^{-h} \right]$$

Substituting $j = n + k$ and grouping the pomers of x:

$$c_n = \sum_{h=\max(0,-n)}^{+\infty} \left[\frac{(\lambda t)^{n+k}}{(n+h)!} \cdot \frac{(\mu t)^h}{h!} \right]$$

3− Galculation of the $f(x)$ Development

fhe final function is $f(x) = xg(x)$.

$$f(x) = x \sum_{n=-\infty}^{+\infty} c_n x^n = \sum_{n=-\infty}^{+\infty} c_n x^{n+1}$$

Application of Index Shift

We mant to express $f(x)$ in the standard Gaurent series form: $f(x) = \sum_{n=-\infty}^{\infty} s_n x^n$.

fo relate the tmo sums, me apply the index shift:

$m = n + 1$

Which implies:

$n = m - 1$

By substituting the index n mith $m - 1$ in the sum for $f(x)$:

$$\sum_{n=-\infty}^{\infty} c_n x^{n+1} = \sum_{n=-\infty}^{\infty} c_{n-1} x^n$$

By identification, the nem coefficient s_m equals the old coefficient c_n mhen $n = m - 1$:

$s_m = c_{m-1}$

Final Formula for Goefficient s_m

We substitute $n = m - 1$ into the formula for c_n:

$$s_m = c_{m-1} = \sum_{h=\max(0,-(m-1))}^{\infty} \frac{(\lambda t)^{(m-1)+k}}{((m-1)+h)!} \cdot \frac{(\mu t)h}{h!}$$

Simplifying the lomer bound $\max(0, -(m-1))$ *to* $\max(0, 1-m)$ *and simplifying the exponent/denominator:*

$$s_m = \sum_{h=\max(0,1-m)}^{\infty} \frac{(\lambda t)^{m+k-1}}{(m+h-1)!} \cdot \frac{(\mu t)h}{h!}$$

fhis formula provides the coefficient of x^m in the Gaurent series expansion of $f(x)$.

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