Impact Factor 6.1

Journal of Cyber Security

ISSN:2096-1146

Scopus

DOI

Google Scholar

More Information
www.journalcybersecurity.com

Crossref doi Google Scholar scopus

ON MANNHEIM CURVES IN A STRICT WALKER 3-MANIFOLD

RÉNOVAT NKUNZIMANA AND CHRISTOPHE MBABARE NGIRENTE

Abstract. In this paper we study the geometry of Mannheim curves in a strict Walker 3-manifold and we obtain explicit parametric equations for Mannheim curves and timelike Mannheim curves, respectively. We determine the distance between two corresponding points of the Mannheim pair of curves and show that distance depending of the curvature. We discuss the relationship between the curvature and torsion of a pair of Mannheim curves in a strict Walker Manifold. We finish by an example of Mannheim pair curves for illustrate the result.

1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are the very interesting and important problem. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. For the Bertrand curve α, it shares the normal lines with another curve β, called Bertrand mate or Bertrand partner curve of α. In this paper, we are concerned with another kind of associated curves, called Mannheim curve and Mannheim mate (partner curve) in history of differential geometry. In this work, we call them simply as Mannheim pair.

From the elementary differential geometry we know clearly about the characterizations of Bertrand pair. But there are rather few works on Mannheim pair. According to [6] it is just known that a space curve in $\mathbb{R}^3$ is a Mannheim curve if and only if its curvature κ and torsion κ satisfy the formula $\kappa = \lambda(\kappa^2 + \tau^2)$, where λ is a nonzero constant. In [10], B. Y. Chen characterizes the curve which satisfies $\frac{\tau}{\kappa} = as + b, a \neq 0$. Here, our examples will give the curve which satisfies $\frac{\tau}{\kappa} = sinh(s)$.

In [6], the autors give the neccesary and sufficient conditions for a curve in 3 Euclidean space to be a Mannheim partner of a giveen curve. They show also that the Mannheim curve of generalized helix is a straight line. In [5], the authors proved that the distance between corresponding points of the Mannheim partner curves in three dimensional Heisenberg group is is constant.

Motivated by the above works, in this paper, we study the Mannheim partner curves in three dimensional Walker manifold M^3. We will give the necessary and sufficient conditions for a curve to be a Mannheim partner curve of an other curve in three Walker Manifold. We detrmine the Mannheim partner of the generalized and the slant helices.

The paper is organised as follow: in section 2, we give some preliminaries tools about

2010 *Mathematics Subject Classification.* Primary 53A10; Secondary 53C42, 53C50.

Key words and phrases. Mannheim curve, curvature, torsion, Mannheim partner, Walker manifolds.

Mannheim curves Walker 3-dimensional space. In section 3, we study Mannheim curves in a strict Walker 3-manifolds and the last section talks about Mannheim partner of helices.

2. Preliminaries

2.1. Mannheim partner curves.

Definition 2.1. [6] *Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space with the standard inner product. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, and β a Mannheim partner curve of α. The pair $\{\alpha, \beta\}$ is said to be a Mannheim pair.*

The curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ in 3-dimensional Euclidean space is parametrized by the arc-length parameter s and from definition above the Mannheim partner curve of α is given by $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$ in 3-dimensional Euclidean space $\mathbb{R}^3$ with the help of Figure 1 such that

$$\beta(s) = \alpha(s) + \lambda(s)B(s);\ s \in I$$

where λ is a smooth function on I and B is the binormal vector field of α. We should remark that the parameter s generally is not an arc-length parameter of β.

2.2. The geometry of Walker manifold. A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker ([3]). Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g_f^ϵ) with coordinates (x, y, z) is expressed as

$$g_f^\epsilon = dx \circ dz + \epsilon dy^2 + f(x, y, z)dz^2 \tag{2.1}$$

and its matrix form as

$$g_f^\epsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix} \quad \text{with inverse} \quad (g_f^\epsilon)^{-1} = \begin{pmatrix} -f & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for some function $f(x, y, z)$, where $\epsilon = \pm 1$ and thus $D = \text{Span}\partial_x$ as the parallel degenerate line field. Notice that when $\epsilon = 1$ and $\epsilon = -1$ the Walker manifold has signature $(2, 1)$ and $(1, 2)$ respectively, and therefore is Lorentzian in both cases.

It follows after a straightforward calculation that the Levi-Civita connection of any metric (2.1) is given by [2]:

$$\begin{aligned} \nabla_{\partial_x}\partial z &= \frac{1}{2}f_x\partial_x, \quad \nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \\ \nabla_{\partial_z}\partial z &= \frac{1}{2}(ff_x + f_z)\partial_x + \frac{1}{2}f_y\partial_y - \frac{1}{2}f_x\partial_z \end{aligned} \tag{2.2}$$

where ∂_x, ∂_y and ∂_z are the coordinate vector fields $\frac{\partial}{\partial_x}$, $\frac{\partial}{\partial_y}$ and $\frac{\partial}{\partial_z}$, respectively. Hence, if (M, g_f^ϵ) is a strict Walker manifolds i.e., $f(x, y, z) = f(y, z)$, then the associated Levi-Civita connection satisfies [2]

$$\nabla_{\partial_y}\partial z = \frac{1}{2}f_y\partial_x, \quad \nabla_{\partial_z}\partial z = \frac{1}{2}f_z\partial_x - \frac{\epsilon}{2}f_y\partial_y. \tag{2.3}$$

Note that the existence of a null parallel vector field (i.e $f = f(y,z)$) simplifies the non-zero components of the Christoffel symbols and the curvature tensor of the metric g_f^ϵ as follows:

$$\Gamma_{23}^1 = \Gamma_{32}^1 = \frac{1}{2} f_y,\ \Gamma_{33}^1 = \frac{1}{2} f_z,\ \Gamma_{33}^2 = -\frac{\epsilon}{2} f_y \tag{2.4}$$

Proposition 2.2. *Starting from local coordinates (x, y, z) for which (2.1) holds, Let*

$$e_1 = \partial_y,\ \ e_2 = \frac{2-f}{2\sqrt{2}}\partial_x + \frac{1}{\sqrt{2}}\partial_z,\ \ e_3 = \frac{2+f}{2\sqrt{2}}\partial_x - \frac{1}{\sqrt{2}}\partial_z \tag{2.5}$$

Then they formed a local pseudo-orthonormal frame fields on (M, g_f^ϵ).

Proof. Indeed, we get $g_f^\epsilon(e_1, e_1) = \epsilon$, $g_f^\epsilon(e_2, e_2) = 1$ and $g_f^\epsilon(e_3, e_3) = -1$. □

Let now u and v be two vectors in M. Denoted by $(\vec{i}, \vec{j}, \vec{k})$ the canonical frame in $\mathbb{R}^3$.

Proposition 2.3. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g defined above, we have*

$$\nabla_{e_i} e_j = \begin{bmatrix} 0 & \frac{1}{4} f_y (e_2 + e_3) & -\frac{1}{4} f_y (e_2 + e_3) \\ \frac{1}{4} f_y (e_2 + e_3) & -\frac{\epsilon}{4} f_y e_1 & \frac{\epsilon}{4} f_y e_1 \\ -\frac{1}{4} f_y (e_2 + e_3) & \frac{\epsilon}{4} f_y e_1 & -\frac{\epsilon}{4} f_y e_1 \end{bmatrix}. \tag{2.6}$$

Proof. The curvature tensor field of ∇ is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where $X, Y, Z \in \Gamma(M)$. If we denote by

$$R_{ijk} = R(e_i, e_j)e_k,$$

where the indices i, j, k take the values $1, 2, 3$. Then the non-zero components of the curvature tensor field are [2]

$$R_{121} = -R_{131} = -\frac{1}{4} f_{yy} (e_2 + e_3),$$

$$R_{122} = -R_{123} = -R_{132} = R_{133} = \frac{\epsilon}{4} f_{yy} e_1. \tag{2.7}$$

□

The vector product of u and v in (M, g_f^ϵ) with respect to the metric g_f^ϵ is the vector denoted by $u \times_f v$ in M defined by

$$g_f^\epsilon(u \times_f v, w) = \det(u, v, w) \tag{2.8}$$

for all vector w in M, where $\det(u, v, w)$ is the determinant function associated to the canonical basis of $\mathbb{R}^3$.

Proposition 2.4. *If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two vectors in $\mathbb{R}^{\nvDash}$ then by using (2.8), we have:*

$$u \times_f v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) \vec{i} - \epsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{k} \tag{2.9}$$

Proof. We develop the two member of equation and after a simple calculation using the determinant function we get the resultats. □

Proposition 2.5. [2] *The Walker cross product in M has the following properties:*

1. *The Walker cross product is bilinear and anti-symmetric.*
2. *$X \times_f Y$ is perpendicular both of X and Y .*
3. *The frame defined in (2.5) verify the following: $e_1 \times_f e_2 = -e_3$, $e_2 \times_f e_3 = -e_1$ and $e_3 \times_f e_1 = e_2$.*

Proof. We use the definition of cross product and compute. □

3. Mannheim curves in strict Walker 3-manifold

Let $\alpha : I \subset \mathbb{R} \longrightarrow (M, g_f^{\epsilon})$ be a curve parametrized by its arc-length s.
The Frenet frame of α is formed by the vectors T, N and B along α where T is the tangent, N the principal normal and B the binormal vector.

Theorem 3.1. [2] *They satisfied the Frenet formulas*

$$\begin{cases} \nabla_T T(s) &= \epsilon_2 \kappa(s) N(s) \\ \nabla_T N(s) &= -\epsilon_1 \kappa T(s) - \epsilon_3 \tau B(s) \\ \nabla_T B(s) &= \epsilon_2 \tau(s) N(s) \end{cases} \tag{3.1}$$

where κ and τ are respectively the curvature and the torsion of the curve α, with $\epsilon_1 = g_f(T;T); \epsilon_2 = g_f(N;N)$ and $\epsilon_3 = g_f(B,B)$.

Proof. We can consider the unit speed normal which is opposite of the principal normal vector. □

Theorem 3.2. *For a Mannheim curve α thre exists a Mannheim partner β such that $\{\alpha, \beta\}$ is a pair of Mannheim curves.*

Proof. As N and B^* are linearly dependants,

$$\begin{aligned} \beta &= \alpha - \lambda B^* \\ &= \alpha - \lambda k N. \end{aligned} \tag{3.2}$$

□

Theorem 3.3. *Let (α, β) be a Mannheim pair in Walker manifold M. The distance between corresponding points of the Mannheim partner curves in M is constant if and only if the curvature of α is a constant.*

Proof. Let $\{\alpha, \beta\}$ be a couple of Mannheim curves in a strict Walker 3-manifold. We note $\{T, N, B\}$ and $\{T^\star, N^\star, B^\star\}$ the Frenet frames of the curves α and β respectively. According to the figure , we can write :

$$\alpha(s) = \beta(s^\star) + \lambda(s^\star) B^\star(s^\star) \tag{3.3}$$

By derivation of equation 3.3 we have :

$$\frac{d\alpha(s)}{ds}\frac{ds}{ds^\star} = \frac{d\beta(s^\star)}{ds^\star} + \frac{d\lambda(s^\star)}{ds^\star} B^\star(s^\star) + \lambda(s^\star)\frac{dB^\star(s^\star)}{ds^\star} \tag{3.4}$$

Using the above equation 3.4 and the fact of N et B^* coincide, and $B^* = kN$ we have :

$$T(s)\frac{ds}{ds^\star} = T^\star(s^\star) + \lambda^{'}(s^\star) kN(s) + \lambda(s^\star)\varepsilon_2^* \tau^\star(s^\star) N^\star(s^\star) \tag{3.5}$$

As N and $B^\star$ are linearly dependent, $g(T, B^\star) = k.g(T, N) = 0$ and we have : $\lambda^{'}(s^\star) = 0$ and in that case λ is a non zero constant. On ather hand, according to the definition of distance function between tzo points we get :

$$\begin{aligned} d\left(\beta\left(s^{\star}\right), \alpha\left(s\right)\right) &= ||\alpha\left(s\right) - \beta\left(s^{\star}\right)|| \\ &= |\lambda\left(s^{\star}\right)|\ ||B^{\star}\left(s^{\star}\right)|| \\ &= |\lambda|\sqrt{\langle B^{\star}, B^{\star}\rangle} \\ &= |\lambda|\sqrt{\langle kN\left(s\right), kN\left(s\right)\rangle} \\ &= |\lambda||k|\sqrt{|g_f^{\varepsilon}\left(N, N\right)|} \\ &= |\lambda||k|\sqrt{|\varepsilon_2|}. \end{aligned}$$

So the distance $d\left(\beta\left(s^{\star}\right), \alpha\left(s\right)\right)$ is constant if k is constant. □

We etablish now the relation between curvatures and torsions of α and β at the correspondants points.

Theorem 3.4. *Let $\{\alpha, \beta\}$ be a pair of Mannheim in Walker 3-manifold. Then the torsion of β is obtained as*

$$\tau^* = \frac{\varepsilon_1}{\varepsilon_3\varepsilon_2^*}.\frac{\kappa}{\lambda\tau}$$

Proof. According to the relation $T\dfrac{ds}{ds^*} = T^* + \varepsilon_2^*\lambda\tau^*N^*$. we get

$$T = \frac{ds^*}{ds}T^* + \varepsilon_2^*\lambda\tau^*\frac{ds^*}{ds}N^*. \tag{3.6}$$

And we have

$$\begin{cases} T &= cos\theta T^* + sin\theta N^* \\ B &= -sin\theta T^* + cos\theta N^* \end{cases} \tag{3.7}$$

where θ is the angle between T and T^* at the corresponding points of α and β respectively.

From (3.6) et (3.7), we have

$$cos\theta = \frac{ds^*}{ds}, \tag{3.8}$$

$$sin\theta = \varepsilon_2^*\lambda\tau^*\frac{ds^*}{ds}. \tag{3.9}$$

By derivation of (3.2), we obtain :

$$\begin{aligned} \frac{d\beta\left(s^*\right)}{ds^*} &= \frac{d\alpha\left(s\right)}{ds}\frac{ds}{ds^*} - k\lambda\frac{dN}{ds}\frac{ds}{ds^*} \\ \Rightarrow T^* &= T\frac{ds}{ds^*} + k\varepsilon_1\lambda\kappa T\frac{ds}{ds^*} + k\varepsilon_3\lambda\tau B\frac{ds}{ds^*}, \end{aligned}$$

$$T^* = \Big(1 + k\varepsilon_1\lambda\kappa\Big)\frac{ds}{ds^*}T + k\varepsilon_3\lambda\tau\frac{ds}{ds^*}B. \tag{3.10}$$

The equation (3.7) give

$$\begin{cases} cos\theta T^* = T - sin\theta N^* \\ sin\theta T^* = -B + cos\theta N^* \end{cases}$$

$$\Rightarrow \begin{cases} T^* = \dfrac{T - sin\theta N^*}{cos\theta} \\ T^* = \dfrac{-B + cos\theta N^*}{sin\theta}. \end{cases} \tag{3.11}$$

The equation

$$3.11 \Rightarrow \frac{T - sin\theta N^*}{cos\theta} = \frac{-B + cos\theta N^*}{sin\theta}$$

$$\Rightarrow sin\theta\left(T - sin\theta N^*\right) = cos\theta\left(-B + cos\theta N^*\right)$$

$$\Rightarrow sin\theta T - sin^2\theta N^* = -Bcos\theta + cos^2\theta N^*$$

$$\Rightarrow sin\theta T + Bcos\theta = \left(cos^2\theta + sin^2\theta\right) N^*.$$

According to $cos^2\theta + sin^2\theta = 1$, we get the equation :

$$N^* = sin\theta T + Bcos\theta,$$

$$\Rightarrow N^* = sin\theta T + cos\theta B.$$

And from the equation 3.7, we obtain :

$$\begin{cases} sin\theta N^* = T - cos\theta T^* \\ cos\theta N^* = B + sin\theta T^* \end{cases}$$

$$\Rightarrow \begin{cases} N^* = \dfrac{T - cos\theta T^*}{sin\theta} \\ N^* = \dfrac{B + sin\theta N^*}{cos\theta} \end{cases} \tag{3.12}$$

$$3.12 \Rightarrow \frac{T - cos\theta T^*}{sin\theta} = \frac{B + sin\theta N^*}{cos\theta}$$

$$\Rightarrow cos\theta\left(T - cos\theta T^*\right) = sin\theta\left(B + sin\theta T^*\right)$$

$$\Rightarrow cos\theta T - cos^2\theta T^* = sin\theta B + sin^2\theta T^*$$

$$\Rightarrow cos\theta T - sin\theta B = \left(sin^2\theta + cos^2\theta\right) T^*.$$

Using $\left(sin^2\theta + cos^2\theta\right) = 1$, we have

$$T^* = cos\theta T - sin\theta B. \tag{3.13}$$

D'o

$$\begin{cases} T^* & = cos\theta T - sin\theta B \\ N^* & = sin\theta T + cos\theta B. \end{cases} \tag{3.14}$$

From (3.10) and (3.13), we obtain :

$$cos\theta = \left(1 + k\varepsilon_1\lambda\kappa\right)\frac{ds}{ds^*}, \tag{3.15}$$

$$sin\theta = -k\varepsilon_3\lambda\tau\frac{ds}{ds^*}. \tag{3.16}$$

By multiplication of the two equation (3.8) and (3.15), and (3.9) and (3.16) respectively we get

$$\begin{aligned} cos^2\theta &= 1 + k\varepsilon_1\lambda\kappa, \\ sin^2\theta &= -k\varepsilon_3\varepsilon_2^*\lambda^2\tau\tau^*. \end{aligned} \tag{3.17}$$

Adding the equation (3.17) we have

$$cos^2\theta + sin^2\theta = 1 + k\varepsilon_1\lambda\kappa - k\varepsilon_3\varepsilon_2^*\lambda^2\tau\tau^*.$$

And from $cos^2\theta + sin^2\theta = 1$ we have

$$\begin{aligned} 1 + k\varepsilon_1\lambda\kappa - k\varepsilon_3\varepsilon_2^*\lambda^2\tau\tau^* &= 1 \\ \Rightarrow -k\varepsilon_3\varepsilon_2^*\lambda^2\tau\tau^* &= 1 - 1 - k\varepsilon_1\lambda\kappa \\ \Rightarrow \tau^* &= \frac{\varepsilon_1\lambda\kappa}{\varepsilon_3\varepsilon_2^*\lambda^2\tau} \\ \Rightarrow \tau^* &= \frac{\varepsilon_1\kappa}{\varepsilon_3\varepsilon_2^*\lambda\tau}, \end{aligned}$$

So we have $\tau^* = \frac{\varepsilon_1}{\varepsilon_3\varepsilon_2^*}.\frac{\kappa}{\lambda\tau}$ □

Theorem 3.5. *Let $\{\alpha, \beta\}$ be a pair of Mannheim in Walker manifold. We have*

$$\varepsilon_3\mu\tau - \varepsilon_1\lambda\kappa = \frac{1}{k},$$

where λ and μ are nonzero real numbers.

Proof. We use the fact that

$$cos\theta = \Big(1 + k\varepsilon_1\lambda\kappa\Big)\frac{ds}{ds^*}$$

et

$$sin\theta = -k\varepsilon_3\lambda\tau\frac{ds}{ds^*}$$

$$\Rightarrow \frac{cos\theta}{1 + k\varepsilon_1\lambda\kappa} = \frac{ds}{ds^*}, \tag{3.18}$$

and

$$-\frac{sin\theta}{k\varepsilon_3\lambda\tau} = \frac{ds}{ds^*}. \tag{3.19}$$

Adding the relations (3.18) and (3.19); and after calculation we get the result □

Journal of Cyber Security(2096-1146) || Volume 7 Issue 12 2025 || www.journalcybersecurity.com

4. Conclusion

In this paper we study the geometry of Mannheim curves in a strict Walker 3-manifold. In the first time we introduced the geometric elements of the strict Walker 3-manifold by calculation of the Christoffel symbols, the Levi-Civita connection, curvature and the cross product. The second concerned our results. In this paper, two main results are obtained. The first one is that in contrast of the case of Euclidean, the distanance between two corresponding points is nit constant. In the second resultat we etablished the relation between the torsion of the partner and the curvature and torsion of Mannheim curve. This paper show that some resultats in Euclidean space can be generalized in the Walker manifold. In the future, we can extend our study to the Mannheim curves in the Walker 4-manifolds.

References

1. B. O'Neill . *Semi-riemannian geometry.* Department of mathematics University of California Los Angeles, California, 1983.
2. M. Gningue, A. Ndiaye and R. Nkunzimana, *Biharmonic Curves in a Strict Walker 3-Manifold*. Int. J. Math. Math. Sci, Volume 2022, Article ID 3855033, 6 pages https://doi.org/10.1155/2022/3855033
3. M. Brozos-Vázquez, E. García-Rio, P. Gilkey, S. Nikević and R. Vázquez-Lorenzo. The Geometry of Walker Manifolds, Synthesis Lectures on Mathematics and Statistics, 5. Morgan and Claypool Publishers, Williston, VT, 2009.
4. M. P. do Carmo, Differential geometry of curves and surfaces, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976. viii+503 pp. 190-191.
5. Sidika Tul and Ayhan Sarioglugil, *Mannheim partner curves in 3-dimensional Heisenberg group*, International Journal of Physical Sciences (2011), Vol. 6(36), pp. 8093-8096.
6. Fan Wang and Huili Liu, *Mannheim Partner Curves in 3-space*, Proceedings of The eleventh International Workshop on Diff. Geom. 11 (2007), pp. 25-31.
7. Handan Balgetir, Mehmet Bektas and Jun-ichi Inoguchi, *Null Bertrand curves in Minkowski 3-space and their characterizations*, Note di Matematica (2004) Vol. 23, no. 1, pp. 7-13.
8. Alexander YAMPOLSKY, Anastasiya OPARIY, *Generalized helices in three-dimensional Lie groups*, Turk. J. Math. (2019) 43, 1447-1455.
9. Shun'ichi HONDA and Masatomo TAKAHASHI, *Bertrand and Mannheim curves of framed curves in the 3-dimensional Euclidean space* , Turk. J. Math. (2020) 44, 883-899.
10. B.Y. Chen, , *When Does the Position Vector of a Space Curve Always Lie in Its Rectifying Plane?*, Amer. Math. Monthly 110 (2003) 147-152.

Rénovat Nkunzimana
Département de Mathématiques, Faculté des Sciences,
Université du Burundi,
B.P 2700, Bujumbura, Burundi.
E-mail address: renovat.nkunzimana@ub.edu.bi

Christophe Mbabare Ngirente
Département de Mathématiques, UAGO,
Université Adventiste de Goma,
B.P 116, Goma, RD Congo.
E-mail address: mbabarechristophe@gmail.com