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On the model foliations of pseudo-Anosov bundles over circle which fibers are surfaces of genus $g \geq 2$

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Abstract

In this paper we construct the models foliations on pseudo Anosov bundle using the Gyhs Thurston's method which consists of the suspension of diffeomorphism followed by desingularization and we study some properties of those foliations.

1 Introduction

In [3] E. Ghys and V. Sergiescu showed that if F is a C^r, $r \geq 2$ codimension 1 transversely orientable foliation without compact leaf on a T^2-bundle $\pi : V \longrightarrow S^1$ over S^1 , then F is C^{r-2} conjugate to one of the model foliations constructed on that bundle. The same autors chowed that the models foliations are structurally stable. Let ψ be an Anosov-diffeomorphism of the torus T^2 induced by a matrix $A \in SL(2; \mathbb{Z})$ with $trA > 2$. E. Ghys and V. Sergiescu (see [3]) built the 3 manifold T^3_A by suspension of ψ and classified all $C^r(r \geq 2)$ codimension 1 foliations without compact leaves.
Let Σ be the genus 2 closed surface. Using the method of branched covering kindly chosen, we obtained from a pseudo-Anosov diffeomorphism φ of Σ. The 3 manifold V_φ^3 is built by suspension of φ. Unlike T_A^3, there is no completed theorem of classification of codimension 1 foliations without compact leaves on V^3_φ. If F is a foliation without compact leaves on a pseudo- Anosov bundle V^3_φ the main dificulty to get a complet classification theorem is because it is dificult as in an Anosov bundle to get a fiber S so that the singular foliation $\mathsf{F}_{|(S\times\{0\})}$ coincides with $\mathsf{F}_{|(S\times\{1\})}$ and to classify the foliation of $\mathsf{F}_{|(S\times[0;1])}$. There are partial results in this direction. For example H. Nakayama (see [6]) classified up to covering, all transversely affine foliation without compact leaves and in the Euler class of the fibration. In the present paper we extend the construction of model foliations on a pseudo Anosov 3 bundles which fibers are surfaces of genus. We note $g \geq 2$ V^3_φ a such bundle. We give some properties of those models foliations in particular we show that those models are close to the fibration and so have the same Euler class as the fibration. Our paper is organised as follow. In the first two paragraphs we gave the construction of

models foliations on some Anosov bundle over S^1 and models foliations on some pseudo-Anosov bundle over S^1 to understand the topics. In the last paragraph we analyse some properties of models foliations and foliations without compact leaves on V^3_φ which are in improved general position.

Indeed, in [10] we have classified, up monotonic equivelence, all taut foliations on a pseudo-Anosov bundle which is hyperbolic and which have the same Euler class as fibration. Here we show that every foliations without compact leaf and which is improved general position on 3 pseudo Anosov bundles, is conguate to the models foliations.

2 Preliminaries

When we put together all the maximal integral manifolds of a p-dimensional involutive distribution D, we obtain a decomposition of M into p-dimensional submanifolds that fit together locally like the slices in a flat chart. We define a foliation of dimension p or codimension $n - p$ on a n-manifold M to be a collection of disjoint, connected, immersed p-dimensional submanifolds of M, called leaves of foliation, whose union is M and such that in a neighborhood of each point $x \in M$ there is a smooth chart (U, ϕ), called flat chart for the foliation, with the property that $\phi(U)$ is a product of connected open sets $V \times W \in \mathbb{R}^p \times \mathbb{R}^{n-p}$, and each leaf of the foliation intersects U in either the empty set or a countable union of p-dimensional slices of the form $x^{p+1} = c^{p+1}, \cdots, x^n = c^n$. Our study is limited on the foliations of codimension one on the 3-manifolds; it means that the leaves of the foliations are surfaces. Locally, we have some things like that

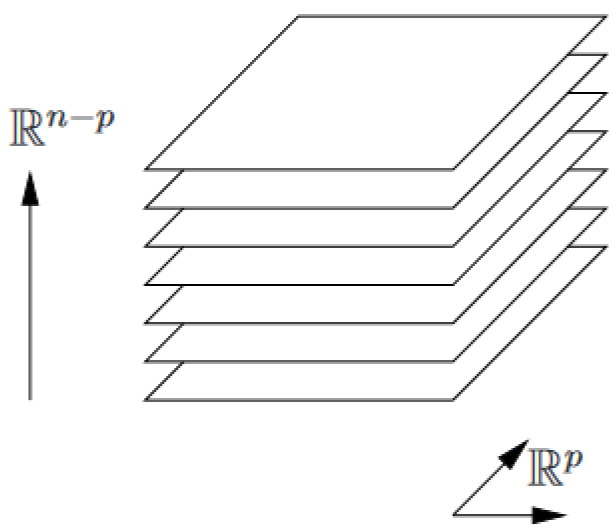

Figure 1: Local representation of a foliation

For example [1] the collection of connected components of the curves in the (y, z)-plane defined by the following equations

$$\begin{cases} z &= sec(y) + c, \ c \in \mathbb{R} \\ y &= (k + \frac{1}{2}), \ k \in \mathbb{Z} \end{cases}$$

is a foliation on a plane.

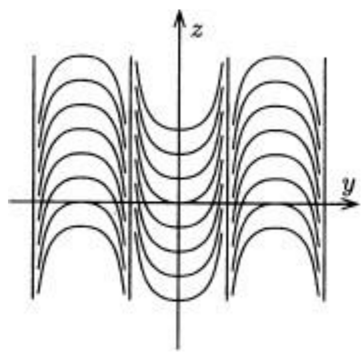

Figure 2: A foliation on $\mathbb{R}^2$

If we rotate the curves of this foliation around the z-axis, we obtain a 2-dimensional foliation of $\mathbb{R}^3$ in which some of the leaves are diffeomorphic to disks and some are diffeomorphic to the cylinders.

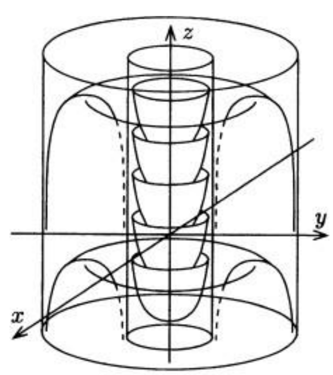

Figure 3: A foliations on $\mathbb{R}^3$

A second example [1], we consider the 2-dimensional torus T^2 with coordinates (x, y). The subbundle K spanned by the vector field $X = \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y}$ with $\alpha \in \mathbb{R}$ is a foliation. If α is irrational, the leaves of K are immersed copies of the line wrapping around the torus infinitely many times, and each leaf is dense. In this case, K is called the Kronecker foliation. And if α is rationnal K is diffeomorphic to a circle.

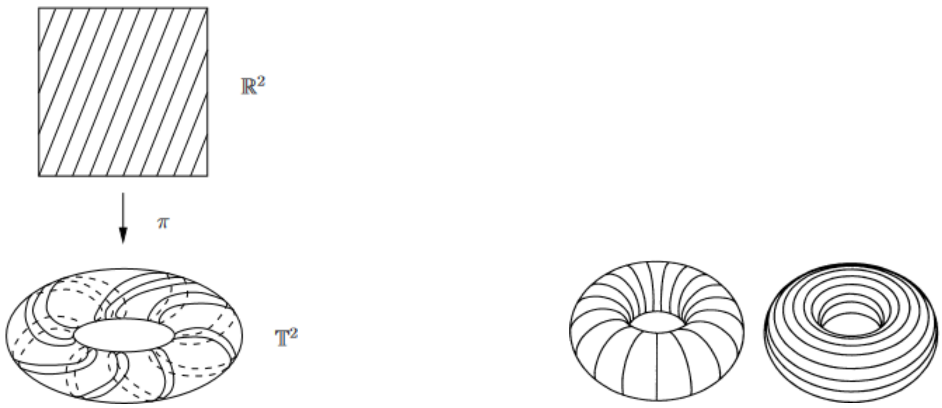

Figure 4: A linear foliation on T^2

A foliation F on a Riemannian manifold (M, g) is called minimal if all its leaves are minimal surfaces in (M, g) (i.e. they locally minimize area; for any compact piece K of a leaf, any small perturbation of K rel ∂K will have bigger area; that is equivalent to each leaf having zero mean curvature). A foliation F on M is called taut if there is a Riemannian metric g such that F is minimal in (M, g). (See [1], ch. 10 for a general discussion.)

Remark 2.1. *In the special case of a codimension 1 foliation, tautness is equivalent to the existence of a dimensional manifold transverse to* F *and crossing all the leaves. A similar condition is too strong for higher codimensions.*

3 Model foliations on pseudo-Anosov bundles

3.1 Elementary foliations on $D^2 \times S^1$

We will construct two foliations on $D^2 \times S^1$ transverse to boundary $\partial(D^2 \times S^1)$ which will serve as surgery of model foliations on the whole bundle.

A foliation whithout singularity on T^2

On $S^1 = [0, 2]/0 \sim 2$ we consider the function f defined by

$$f: \ [0, 2] \longrightarrow \mathrm{R}$$
$$y \longrightarrow z = f(y) = Log \mid sec\, y \mid,$$ together with the vertical lines cos y=0

We have $f^{-1}(0) = \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\}$. We consider on T^2, the foliation F_k defined by the 1-form $\omega_k = df + kfdz$, where (y, z) are the coordonates on R^2 and $T^2 = R^2/Z^2$. The only compact leaves of F_k are $T \times S^1$ where T is a connected component of $f^{-1}(0)$. The compact leaves have the holonomy and the no compact leaves roll up to compact leaves. So F_k is

a foliation with four Reeb plane components (see Figure 1). F_k is a one parameter family of transversely affine foliations with holonomy on T^2.
If we rotate the curves of this foliation around the z-axis, we obtain a 2-dimensional foliation of R^3 in which some of the leaves are diffeomorphic to disks and some are diffeomorphic to the cylinders

A family of regular foliations on $D^2 \times S^1$

Consider the 1-form $\omega_0 = xdx - ydy$ of saddle type on R^2 and the 1-form $\tilde{\Omega}_\lambda = \lambda^z \omega_0 + dz, \lambda \in N^*$ on $R^2 \times R$. It is the nonsingular 1-form satisfying $d\tilde{\Omega}_\lambda = \tilde{\alpha}_\lambda \wedge \tilde{\Omega}_\lambda$ where $\tilde{\alpha}_\lambda = (Log\lambda)dz$. Then $\tilde{\Omega}_\lambda$ defines a transversely affine foliation on $R^2 \times R$. We consider the function

$$\begin{array}{rccc} h_\lambda : & R^2 \times R & \longrightarrow & R^2 \times R \\ & (u, z) & \longrightarrow & (\lambda u, z + 1) \end{array}$$

The quotient by the action generated by h_λ gives a cyclic covering $R^2 \times R \rightarrow R^2 \times S^1$. The orded pair $(\tilde{\Omega}_\lambda, \tilde{\alpha}_\lambda)$ is invariant by this action and passes to the quotient in an orded pair $(\Omega_\lambda, \alpha_\lambda)$ which defines a regular foliation on $R^2 \times S^1$.

3.2 The explicit model foliations on V

Now we can construct the model foliations on V using the Ghys-Thurston method sketched in many works one of which being [6]. The method consists of a suspension of the stable and unstable foliations followed by a desingularization. On $\Sigma \times R$, we consider the 1-form $\Omega^+ = \lambda^t \omega^+ + \psi dt$ where ψ is C^∞ with support in a canonical neighborhood of singularities of ω^+. The form Ω^+ is invariant by the action $(x, t) \sim (\phi(x), t + 1)$ and passes to the quotient $V = \Sigma \times R/\sim$ in a minimal transversely affine and singular foliation H^+ which has a finite number of contact circles $\gamma_1, \ldots, \gamma_n$ with the fibration of V over S^1. According to the last constructions and the lemma below, each circle γ_i has a tubular neighborhood V_i diffeomorphic to $D^2 \times S^1$ foliated as in the figure below (Figure 2), such that ∂V_i is transverse to H^+ and $H^+_{|\partial V_i}$ is the foliation F_k on the torus T^2 described above for $k = Log\lambda$. We define in the same way H^- using ω^-.

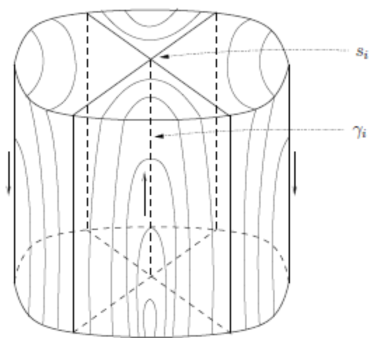

Figure 5: A foliated neigborhood of the singularity [6]

Lemma 3.1. H^+ *or* H^- *is fixed as above.*

a) There exists a neighborhood $M \subset V$ *of* $\{0\} \times \mathrm{S}^1$ *diffeomorphic to* $\mathrm{D}^2 \times \mathrm{S}^1$ *such that the boundary* ∂M *of* M *is transverse to* H^+ *or* H^-.

b) For all M *satisfying* a*), the trace of* H^+ *or* H^- *on* ∂M *is diffeomorphic to foliation* F_k *of* T^2 *for* $k = Log\lambda$ *or* $k = -Log\lambda$.

Proof. Let $\sigma_1, \sigma_2, \sigma_3$ and σ_4 be the separatrices of the saddle singularity 0 of Ω^+. All the leaves $\sigma_i \times \mathrm{S}^1$ of H^+ have holonomy. We can take a tubular neighborhood $M \subset V$ of $0 \times \mathrm{S}^1$ such that $\partial M \simeq \mathrm{T}^2$ is transverse to H^+. The holonomy of the leaves $\sigma_i \times \mathrm{S}^1$ is determined by the linear holonomy $-Log\lambda dz$. Therefore $\mathsf{H}^+{}_{|\partial M}$ is a foliation on torus T^2 with four circular compact leaves having holonomy. The other leaves of $\mathsf{H}^+{}_{|\partial M}$ are not compact and they spiral on the four compact leaves to give a foliation of T^2 with four Reeb plane components and whose compact leaves have $-Log\lambda$ as coefficient of holonomy; $\mathsf{H}^+_{|\partial M}$ is then a foliation of type F_k for $k = Log\lambda$. □

We desingularize the foliations H^+ and H^- as follows:

Let S be the finite singular set common to ω^+ and ω^-. We have constructed above two foliations on $\mathrm{D}^2 \times \mathrm{S}^1$ which are used to complete H^+ and H^- by surgery on whole 3-manifold V. On $(\Sigma - S) \times \mathrm{R}$, the 1-forms $\lambda^t\omega^+ + dt$ and $\lambda^t\omega^+ - dt$ define two nonsingular foliations V_1^+ and V_2^+ on $V - S'$, with $S' = \cup\gamma_{,i}$ which have respectively $\chi(T\pi)$ and $-\chi(T\pi)$ as Euler classes. Each circle γ_i of S has a neighborhood W_i whose boundary is transverse to V_i $(i = 1, 2)$. So we cut the V_i and they are replaced by the W_i foliated as in the figures below. When the transverse orientation of $\mathsf{H}^+|\partial V_i$ is the same as the orientation of γ_i, we replace V_i by W_i with foliation V_1, otherwise V_i is replaced by W_i with foliation V_2. We obtain two regular foliations F^1_+ and $\mathsf{F}^2{}_+$transversely affine without compact leaf on V. Likewise, H^- gives two regular foliations F^1_- and $\mathsf{F}^2{}_-$.

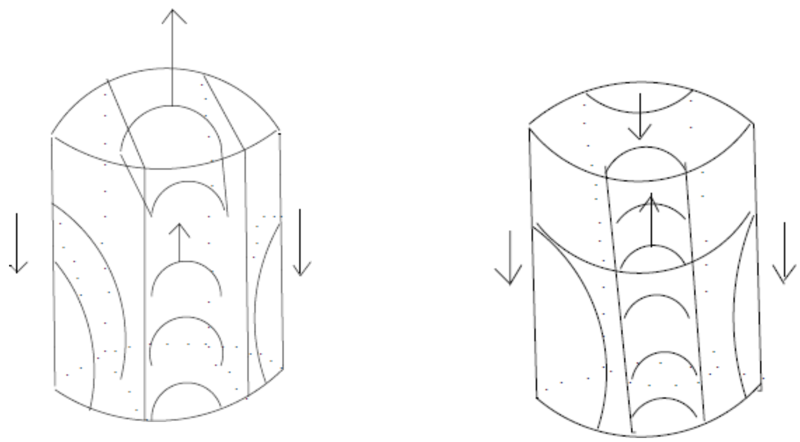

Figure 6: Regular foliation after desingularization

4 Some properties of models foliations

In this section, we give some properties of model foliations constructed above.

Proposition 4.1. *The foliations $\mathcal{F}_k$ are transversely affine foliations and are without holonomy.*

Proof. Indeed, $d\omega_k = kdf \wedge dz = df \wedge kdz$ and $\omega_k \wedge kdz = df \wedge kdz$. Again, if we note $\alpha_k = kdz$, we remark that α_k is closed one forme. Hence $\omega_k = d\omega_k \wedge \alpha_k$ with α_k a closed 1-form [2]. So $\mathcal{F}_k$ is transversely affine foliation for all k real number. □

Proposition 4.2. *The model foliations constructed above having the same Euler class of fibration are close to the fibration.*

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ model foliations of V which have the same Euler class as fibration π. The foliations $\mathcal{F}$ and $\mathcal{F}'$ are defined by the 1-forms $\Omega^0 = \lambda^t\omega^+ + dt$ and $\Omega^1 = \lambda^{-t}\omega^- + dt$. We consider the family of continues one parameter forms $\mathcal{F}^-_m$ and $\mathcal{F}^+_m$ of transversely affines foliations defined by $\Omega^+_m = m\lambda^t\omega^+ + dt$ and $\Omega^-_m = m\lambda^{-t}\omega^+ + dt$. For $m = 0$, $\mathcal{F}^+_m$ and $\mathcal{F}^-_m$ are fibrations for $m > 0$, $\mathcal{F}^+_m$ and $\mathcal{F}^-_m$ have their holonomy representations conjugate to those of $\mathcal{F}$ and $\mathcal{F}'$ [11]. And so are respectively isotopic. □

5 Foliations which are in improved general position

The usual technique in studing foliations of 3-manifolds is to restrict them to their 2 dimensional submanifolds. Unfortunately the foliations induced on those surfaces are not always simple. Indeed they have singularities which are generically of Morse type (i.e saddle or center singularities). In some cases, singularities can be reduced

Theorem 5.1. *Let V be a closed 3-manifold and $\mathcal{F}$ a C^2 taut foliation on V. Let $\rho : \Sigma \to V$ be an embedding of a closed surface Σ in V. We assume that $\rho_* : \pi_1\Sigma \to \pi_1 V$ is injective. Then ρ is isotopic to an embedding ρ' which is tangent to $\mathcal{F}$, or transverse to $\mathcal{F}$ except at a finite number of points which are saddle singularities with an even number of separatrices for $\mathcal{F}$.*

This theorem is proved by Roussarie in the case where the surface Σ is torus T^2 (see [9]). In this paper we use the same technique with a surface of genus $g \geq 2$. Here we show this theorem using a method of Riemannian geometry.

Proof. According to J. Hass [7] there exists a Riemannian metric g making all leaves of $\mathcal{F}$ minimal submanifolds. Every fiber Σ is incompressible and so isotopic to a minimal orientable and closed surface.
Applying Schoen-Yao result [8] we can say that ρ is isotopic to an embedding ρ' which is minimal relatively to g and we use the same result on the intersections of minimal surfaces according to which the trace of a minimal foliation on a minimal surface which is not a leaf is a singular foliation whose saddle singularities have an even number of separatrices. Then, by small isotopies, we can explode the $2n$-branches saddle singularities into 4-branches isolated saddle singularities. □

We give now the following definition:

Definition 5.2. Let $\mathcal{F}$ be a codimension 1 foliation on a hyperbolic bundle $\pi : V \longrightarrow S^1$.

a) We say that $\mathcal{F}$ is in general position with respect to the fibration π if :

   i) there exists a finite family $\Gamma \subset V$ of embedded circles, said braid, such that $T_u\mathcal{F} = T_u\pi$ if and only if $u \in \Gamma$.

   ii) every component γ of Γ has a tubular neighborhood $V(\gamma)$ in which $\mathcal{F}$ is transverse to π except at a finite number of points (restricted to any fiber) which are singularities of type saddle or center type.

b) $\mathcal{F}$ is in improved general position if in ii) we have only saddle singularities.

We have the illustration of this definition at the figure bellow, in the case of fibers are surfaces fo genus $g = 2$ (See Figure 7).

Proposition 5.3. *Let V be a closed 3-manifold bundle over circle S^1 and $\mathcal{F}$ a C^2 taut foliation on V. If $\mathcal{F}$ has no compact leaf, then every fiber of V is isotopic to a closed surface in optimal position (i.e transverse to $\mathcal{F}$ except at a finite number of points which are saddles with 4 separatrices).*

Proof. Indeed, the exact sequence of homotopy associated to the fibration shows that the canonical injection of each fiber of V is injective. Then we apply the theorem above to obtain the points of contact which are saddles with an even numbers of separatrices, by sequence of small isotopies, we can explode all saddles into isolated saddles with 4 separatrices. □

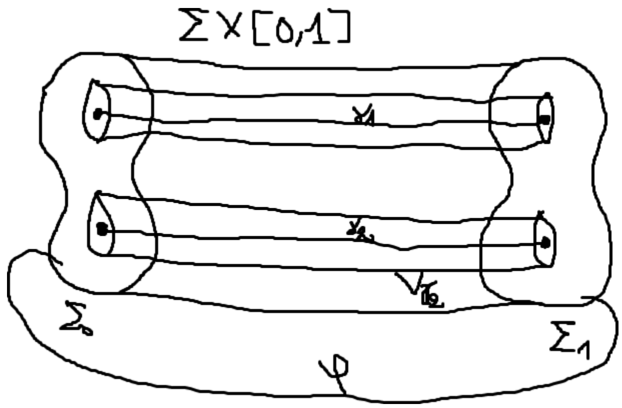

Figure 7: A foliation in optimal position

Corollary 5.4. *The models foliations construct above are in improved general position.*

The following theorem gives a type of classification of taut foliations in improved general position with respect to the fibration on atoroidal 3-manifolds bundles over circle S^1.

Theorem 5.5. *[5] Let F be a codimension 1 taut foliation on a closed 3-manifold which is a hyperbolic bundle $\pi : V \longrightarrow S^1$ over S^1 with monodromy $\phi : \Sigma \to \Sigma$ where Σ is a closed oriented surface of genus $g > 1$ and having the same Euler class as the fibration. Let $\tilde{F}$ be a foliation induced by F on $\Sigma \times [0, 1]$ obtained by cutting V along a fiber Σ. If the foliation F is in improved general position with respect to fibration π, then $\tilde{F}$ is isotopic to a product foliation $\tilde{F_0} \times [0, 1]$ where $\tilde{F_0}$ is isotopic to a caracteristic foliation of ϕ.*

Theorem 5.6. *Any regular foliation F can be put in improved general position about fibration using small isotopy.*

Proof. To prove this theorem we suppose that the foliation F have no compact leaf. According to theorem 5.1 and using lemma bellow.

Lemma 5.7. *After operating a finite number of isotopies, every fiber Σ of V is in optimal position.*

We show also that lemma

Lemma 5.8. *Let Σ_0 be a fiber of V which is in optimal position and consider a diffeomorphism ϕ of monodromy of V. Cutting V along Σ_0 and note $\tilde{F}_0$ and $\tilde{F}_1$ the two foliations induced by F on $\Sigma_0 \times \{0\}$ and $\Sigma_0 \times \{1\}$. Then for every singularity s of $\tilde{F}_0$, there exists a loop $\lambda : [0, 1] \to \Sigma_0 \times [0, 1]$ such that $\lambda(0) = s; \lambda(1) = \phi(s)$ and λ is tranversal to $\tilde{F}_0$.*

Proof. As F is without an interior compact leaf, if s is a singularity of $\tilde{F}_0$, then $\phi(s) = s'$ is a singular point of $\tilde{F}_1$ while $\tilde{F}_1 = \phi_* \tilde{F}_0$. Taking a loop λ linked s and s'; and $\tilde{L}$ an interior leaf of $\Sigma_0 \times [0, 1]$. The leaf $\tilde{L}$ is compacteness.

Let $(u_1, \ldots, u_p)$ be a finite distinguished covering for F. As $\tilde{L}$ is not compact, for a certain u_j we have $s \in u_j$ and there exists two plaques of $\tilde{L}$ in u_j. The we take a suitable transveral J having ends in those two plaques. $J \cap \lambda$ is transversal to F. Modifying λ by an isotopy, we can suppose that λ is transversal to F. □

Lemma 5.9. *There is a loop λ satisfying the conditions of the lemma above such that λ is transversal to the fibration.*

Proof. As foliation which leaves are fiber of the fibration is taut, then we just take λ as the closed tansversal meeting all the leaves. □

Hence the theorem □

6 Concusion

In this paper we studied some properties of a class of foliations without leaf compact on a particular surface bundles over circle which fibers are surfaces of genus 2. We have generalized the important resultat of R. Roussarie regarding bundles wich fibers are 2-torus in optimal position to fibres of genus 2 which are in improved general position. That complete the question asked in [10] which consisted in finding thes contitions for a taut folition to be in impoved general position.

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