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On the automorphism of rational groups

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Abstract

In this work, we consider rational groups containing Z and their automorphism groups. We have established, for two rational groups A and B both containing Z, conditions for the isomorphism $Hom(A, Aut(A)) \cong Hom(B, Aut(B))$ to hold. We have then shown that $Hom(Aut(A), A) \cong Hom(Aut(B), B)$ if and only if $A \cong B$.

Keywords: rational group; homomorphism; automorphism.

1. Introduction.

This article is part of the study of the group $Hom\ (A, B)$ of homomorphisms of two rational groups A and B, which constitutes a special case of problem 30 of Laslo FUCHS posed in 1977.

We consider in the group $Hom\ (A, B)$ that the two rational groups A and B all contain Z and let the group $B = Aut(A)$ be the group of automorphisms of the group A.

The objective of this work is to describe the groups $Hom(A, Aut(A))$ and $Hom(Aut(A), A)$ where A is a rational group containing Z.

We will establish that, for two rational groups A and B all containing Z, $Hom(A, Aut(A)) \cong Hom(B, Aut(B))$ if and only if $P_2(A) = P_2(B)$.

Finally, we will establish, for two rational groups A and B all containing Z, the following equivalence $Hom(Aut(A), A) \cong Hom(Aut(B), B) \Leftrightarrow A \cong B$.

2. Notations

We will denote by:

- Z/nZ : the cyclic group equipped with addition of order n;
- $Aut(A)$: the group of automorphisms of the rational group A containing Z;
- $Hom(A, B)$: the group of homomorphisms of the rational group A containing Z into the rational group B containing Z;
- $A(Z) = \{A: (Z, +) \leq A \leq (Q, +)\}$;
- $\bigoplus$: the direct sum;
- Π: the direct product;
- Z: the set of relative integers;
- N: the set of natural numbers
- P: the set of prime natural numbers;
- $P(A) = \{p \in P: pA = A\}$ for all $A \in A(Z)$;

- $P_2(A) = \emptyset$ if $2A \neq A$ and $P_2(A) \neq \emptyset$ if $2A = A$ for all $A \in A(Z)$.

3. Results

To describe the groups $Hom(A, Aut(A))$ and $Hom(Aut(A), A)$ where $A \in A(Z)$, we will start with the following lemma.

Lemma 3.1 [6]. Let the group be $A \in A(Z)$. Then, $Aut(A) \cong Z/2Z \oplus \underset{|P(A)|}{\oplus} Z$.

Remark 3.1. Let the group be $A \in A(Z)$. Then, $Hom(A, Aut(A)) \cong Hom(A, Z/2Z)$.

Proof. Indeed, by applying the conclusion of Lemma 3.1, we have for any rational group $A \in A(Z)$, $Hom(A, Aut(A)) \cong Hom\left(A, Z/2Z \oplus \underset{|P(A)|}{\oplus} Z\right) \cong$

$\cong Hom(A, Z/2Z) \oplus Hom\left(A, \underset{|P(A)|}{\oplus} Z\right) \cong Hom(A, Z/2Z)$.■

From this observation, the following consequences arise:

Consequence 3.1 [6]. Let the group $A \in A(Z)$. Then, $Hom(A, Aut(A)) = \{0\}$ if and only if $P_2(A) \neq \emptyset$. ■

Consequence 3.2. Let the group $A \in A(Z)$. Then, the group $Hom(A, Aut(A))$ is not isomorphic to the rational group A. ■

For two rational groups $A, B \in A(Z)$, we have given conditions for the following isomorphism $Hom(A, Aut(A)) \cong Hom\big(B, Aut(B)\big)$ to occur in this theorem.

Theorem 3.1. Let the groups $A, B \in A(Z)$.

Then, $Hom(A, Aut(A)) \cong Hom\big(B, Aut(B)\big)$ if and only if $P_2(A) = P_2(B)$.

Proof. Indeed, by applying the conclusion of Remark 3.1 above, we obtain two cases:

Case 1: The groups A and B are both 2-divisible, we obtain:

$Hom(A, Aut(A)) \cong Hom(A, Z/2Z) = \{0\}$ if and only if $2A = A \Rightarrow P_2(A) = \{2\}$ and
$Hom(B, Aut(B)) \cong Hom(B, Z/2Z) = \{0\}$ if and only if $2B = B \Rightarrow P_2(B) = \{2\}$.

So $P_2(A) = P_2(B)$.

Case 2: Groups A and B are not both 2-divisible, we obtain:

$Hom(A, Aut(A)) \cong Hom(A, Z/2Z) \cong Z/2Z$ if and only $2A \neq A \Rightarrow P_2(A) = \emptyset$ and
$Hom(B, Aut(B)) \cong Hom(B, Z/2Z) \cong Z/2Z$ if and only if $2B \neq B \Rightarrow P_2(B) = \emptyset$.

Therefore $P_2(A) = P_2(B)$.

Ultimately, $Hom(A, Aut(A)) \cong Hom(B, Aut(B))$ if and only if the sets $P_2(A)$ and $P_2(B)$ are equal. ■

Proposition 3.1. Let A be an element of $A(Z)$.

Then, $Hom(Aut(A), A) \cong \prod_{|P(A)|} A$.

Proof. Indeed, $Hom(Aut(A), A) \cong Hom\left(Z/2Z \oplus \bigoplus_{|P(A)|} Z, A\right) \cong Hom(Z/2Z, A) \oplus Hom\left(\bigoplus_{|P(A)|} Z, A\right) \cong Hom\left(\bigoplus_{|P(A)|} Z, A\right)$ because the rational group $A \in A(Z)$ is torsion-free, so $Hom(A, Z/2Z) = \{0\}$.

Therefore, we obtain:

$Hom(Aut(A), A) \cong Hom\left(\bigoplus_{|P(A)|} Z, A\right) \cong \prod_{|P(A)|} Hom(Z, A) \cong \prod_{|P(A)|} A$. ■

Corollary 3.1. Let A be an element of $A(Z)$.

Then, $Hom(Aut(A), A) \cong A$ if and only if $|P(A)| = 1$. ■

Theorem 3.2. Let $A, B \in A(Z)$, then the following statements are equivalent:

1) $Hom(Aut(A), A) \cong Hom(Aut(B), B)$

2) $A \cong B$.

Proof. 1) $\Leftrightarrow$2) Indeed, by applying the conclusion of proposition 3.1, for all rational groups $A, B \in A(Z)$, we have:

$Hom(Aut(A), A) \cong \prod_{|P(A)|} A$ and $Hom(Aut(B), B) \cong \prod_{|P(B)|} B$.

Retaining that the rational groups $A, B \in A(Z)$are torsion-free and of rank one, we then obtain $\prod_{|P(A)|} A \cong \prod_{|P(B)|} B$ if and only if $A \cong B$.■

4. Conclusion.

In this work, we studied the group of homomorphisms of rational groups containing Z.

For two rational groups $A, B \in A(Z)$, we showed that $Hom(A, Aut(A)) \cong Hom(B, Aut(B))$ if and only if $P_2(A) = P_2(B)$.

We established that, for any rational group $A \in A(Z)$, $Hom(Aut(A), A) \cong A$ if and only if $|P(A)| = 1$. Finally, we showed that the following isomorphism holds: $Hom(Aut(A), A) \cong Hom(Aut(B), B)$ if and only if the groups A and B are isomorphic.

We could consider the group $Hom(A, B)$ of homomorphisms of two rational groups all containing Z by answering the following questions:

1) When is $Hom(A, B)$ isomorphic to A or B?

2) Describe all rational groups containing Z such that $Hom(A, A)$ is isomorphic to A?

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Author Contributions

All authors contributed significantly to the writing of this article. They have read and approved the final manuscript.

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